Algebraic Tiling

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August 3, 2003

We will investigate tilings of the infinite (in both directions) cellular line (in problems 1 and 2 we will also consider tilings on the cellular plane, space etc.).

Definition. A *figure* is any finite set of cells (this set is not supposed to be connected!). It is clear that the area of a figure is a number of its cells (we assume that each cell has area 1). Any translation of a figure (on a vector with integer coordinates) is called a *translate* of this figure. Let $\mathcal{F}$ be a figure. A *$\mathcal{F}$-tiling* is a representation of a cellular line (plane, space etc.) as a disjoint union of translates of $\mathcal{F}$. A figure $\mathcal{F}$ is a *tile* if some $\mathcal{F}$-tiling exists.

A set of cells is a *transversal set* for a figure $\mathcal{F}$ if any translate of $\mathcal{F}$ intersects this set at exactly one cell.

We will also consider functions mapping the cellular line (plane, space) to $\mathbb{C}$, i. e. for each cell there is a complex number corresponding to it (further we will say that the number is *placed* into this cell). Such a function is called a *barrier function* for $\mathcal{F}$ if for any translate of $\mathcal{F}$ the sum of numbers in its cells is zero; a function is called *bounded* if there exists a number M such that absolute values of all numbers in cells do not exceed M.

Some hard problems are marked by a star. The problems which are not necessary for the further advantage in the main stream are marked by $\sharp$.

1. In this row of problems we consider tilings on arbitrary-dimensional cellular space.

1a). Prove that a figure is a tile if and only if there exists a transversal set for it. Moreover, the number of different tilings (maybe infinity) is equal to the number of different transversal sets for this figure.

1b). Prove that for any tile there exists a nontrivial bounded barrier function.

1c). Prove that any tiling on the cellular line is periodic.

1d). Consider a figure on the cellular line and suppose there exists its barrier function with a finite image. Prove that this function is periodic.

Let $\mathcal{F}$ be a figure. A figure symmetric to $\mathcal{F}$ with respect to a point (with coordinates $(a/2, b/2)$, $a, b \in \mathbb{Z}$) is denoted by $\overline{\mathcal{F}}$.

1e)$^\sharp$. Prove that any transversal set for $\mathcal{F}$ is also a transversal set for $\overline{\mathcal{F}}$.

1f)$^\sharp$. (A. Ya. Belov) Prove that the following conditions for a table-napkin are equivalent:

(i) If some its translates are placed on a table, then one can nail them up, any table-napkin being nailed by exactly one nail.

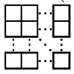
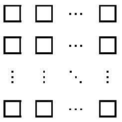
(ii) There exists a plane tiling consisting of translates of this table-napkin.

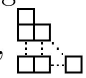
2. Let us enumerate the cells of the cellular line by integers in natural order. Let us assume that the number of the leftmost cell of $\mathcal{F}$ is 0. Associate to $\mathcal{F}$ a polynomial

$$P_{\mathcal{F}}(x) = \sum_{i:\ i\text{th cell is in } \mathcal{F}} x^i.$$

2a). Let ε be a root of the polynomial $P_{\mathcal{F}}$. Consider the function mapping the ith cell to ε^i. Prove that this function is a barrier for $\mathcal{F}$. Such a function will be referred to as a *root function* of $\mathcal{F}$. Try to formulate and prove a similar result for a figure on the cellular plane.

2b)$^\sharp$. A figure is a *n-tile* if some its translates form a tiling of the interval of length n. Prove that the number of different n-tiles does not exceed $2^{d(n)}$ (here, $d(n)$ is the number of divisors of n).

2c)$^\sharp$. We consider a cellular plane. Let the figure $\mathcal{F}$  consist of all the cells with coordinates (a,b): $0 \leqslant a,b \leqslant 2k$; let the figure $\mathcal{G}$  consist of all the cells with coordinates $(2a,2b)$: $0 \leqslant a,b \leqslant 2k$. Prove that any barrier for $\mathcal{F}$ is also a barrier for $\mathcal{G}$.

2d)$^\sharp$. Consider the functions mapping the cellular space to $\mathbb{Z}_2$ (residues modulo 2 with standard addition and multiplication), i. e. each cell has an element of $\mathbb{Z}_2$ placed in it. Find all integers $n > 0$ satisfying the following property: any barrier for the "3-cell corner"  is also a barrier for the "cellular triangle"  with the length of its side equal to n.

All the following problems deal with tilings of the cellular line.

3. A complex number ε is called a *root of unity of degree n* if $\varepsilon^n = 1$. It is a *primitive root of degree n* if n is the least positive integer satisfying this property; in this situation n is called the *primitive degree* of ε. The set of all primitive roots of unity of degree n will be denoted by R_n.

3a). Let (a_k) be a periodic sequence with period n. Prove that it can be represented in the form

$$a_k = \alpha_1\varepsilon_1^k + \alpha_2\varepsilon_2^k + \ldots + \alpha_n\varepsilon_n^k, \tag{1}$$

where ε_i are all different roots of unity of degree n.

3b). Let $\mathcal{F}$ be a figure, (a_k) be its periodic barrier function. Prove that (a_k) can be represented in the form of 1 where ε_i are all roots of unity which are also the roots of the polynomial $P_{\mathcal{F}}$.

3c). Let n_i be all the primitive degrees of the roots of unity which are also the roots of $P_{\mathcal{F}}$. Prove that the period of each $\mathcal{F}$-tiling divides $\mathrm{lcm}(n_i)$ and is a multiple of the area of the tile $\mathcal{F}$.

3d). Solve the problem **1e)** (for the case of the cellular line) using the technique introduced.

4. (Generalization of S.L.Berlov's problem.) Here we will find a criterion for a figure of area p to be a tile (for a prime p). Any figure of area p will be further referred to as a *p-figure.*

Definition. A figure is *incompressible* if the greatest common divisor of the distances between its cells is 1 (informally, this mean that the cellular line cannot be "contracted", our tile lying in this "contraction"). Obviously, it is sufficient to investigate the tilings for incompressible tiles.

4a). Prove that for a given p-figure $\mathcal{F}$ there exists at most one positive integer k such that $P_{\mathcal{F}}$ has roots in R_{p^k}.

4b). If the polynomial of a p-figure has a root in R_{p^k} for $k > 1$ then this figure is not incompressible.

4c)*. Let $\mathcal{F}$ be a tile of area n, where n is a multiple of a prime p. Find the number of such values of k that $P_{\mathcal{F}}$ has roots in R_{p^k}.

4d). Prove that an incompressible p-figure is a tile if and only if the numbers of all its cells have different residues modulo p.

Definition. A *linear combination* of some expressions $X_1,\ldots,X_n$ with rational (complex, integer, etc.) coefficients $a_1,\ldots,a_n$ is the expression $a_1X_1 + \cdots + a_nX_n$.

5. The relation of this problem to the other ones does not seem obvious (but still is important!). We will investigate polynomials $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ (with rational coefficients) having some roots in R_n. Such polynomials will be referred to as *n-root* polynomials.

Let n be a multiple of a prime p, $n=mp$. We call a *p-polygonal polynomial* a polynomial of the form $x^i+x^{i+m}+\dots+x^{i+(p-1)m}$, $i<m$.

5a). Prove that a p-polygonal polynomial is a n-root polynomial.

5b). Prove that any n-root polynomial is a linear combination of some p-polygonal polynomials for some (maybe different) primes dividing n.

5c). Prove that if an n-root polynomial has integer coefficients, then it is a linear combination with integer coefficients of some p-polygonal polynomials.

5d). Prove that if an n-root polynomial has only k nonzero coefficients then it is a linear combination of some p-polygonal polynomials for different $p\leqslant k$.

5e)$^\sharp$. Suppose $\varepsilon_1,\dots,\varepsilon_5$ be roots of unity, $\varepsilon_1+\dots+\varepsilon_5=0$. Prove that these roots are either the vertices of a regular pentagon or the vertices of a regular triangle and two endpoints of a diameter on the complex circle $|z|=1$.

5f)$^\sharp$. (Proposed by K. A. Knop) Describe all the pairs of triangles with coinciding centroids having the vertices among the vertices of a given regular n-polygon.

6. These problems help to prove the following fact: for any incompressible p-tile $\mathcal{F}$ there exists only one $\mathcal{F}$-tiling (up to a translation).

By "different tilings" we mean the tilings that cannot be identified by a translation.

6a). Prove that for any composite $n>1$ there exists an incompressible tile $\mathcal{F}$ of area n with two different $\mathcal{F}$-tilings.

6b). Let $\mathcal{F}$ be a p-tile for which there exist two different $\mathcal{F}$-tilings. Prove that $P_{\mathcal{F}}$ has a root of order $n>p$, n being a multiple of p.

6c). In the case of the previous problem, prove that $\mathcal{F}$ is n/p-times compressible.

The following facts are considered known; you may use them without proof. Nevertheless, if you want you might try to prove them ...

Fact. a). Any polynomial with integer coefficients can be represented as a product of irreducible (over integers) polynomials; this representation is unique up to permutation of factors and multiplication (of some of) them by -1. Any polynomial with complex coefficients may be represented as a product of the form $a(x-x_1)\dots(x-x_n)$; this representation is unique up to permutation of factors.

b). The set of roots of the polynomial

$$F_{p^k}(x)=1+x^{p^{k-1}}+x^{2p^{k-1}}+\dots+x^{(p-1)p^{k-1}}$$

is R_{p^k}. This polynomial is irreducible (over $\mathbb{Z}$). Hence, if a polynomial $P(x)$ with integer coefficients has a root in R_{p^k} then all the elements of R_{p^k} are roots of $P(x)$.

c)*. The polynomial $F_n(x)=\prod_{\varepsilon\in R_n}(x-\varepsilon)$ is irreducible (over $\mathbb{Z}$).