Algebraic tiling. Semifinal

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Corrected and added problems

ATTENTION!!! Problem 6c) is corrected!!!

6c). In the case of the previous problem, prove that $\mathcal{F}$ is either p-times compressible or n/p-times compressible.

7. We consider tilings on the cellular square with side p, the opposite sides being identified (p is prime; such a cellular set will be referred to as the $p\times p$-*torus*). (An equivalent formulation: we consider tilings on the cellular plane s.t. the translation on vector $(0,p)$ as well as on vector $(p,0)$ maps our tiling to itself.)

Let $\mathcal{F}$ be a p-figure on the $p\times p$-torus. Prove that the following conditions are equivalent:

(i) one can put some $\mathcal{F}$-translates on the torus so that each cell of the torus will intersect with exactly k translates ($k<n$);

(ii) there exists an $\mathcal{F}$-tiling on the torus;

(iii) one can paint the cells of the torus in p colours so that for each $\mathcal{F}$-translate all its cells are different-coloured.

Solutions

Let $\mathcal{F}$ be a figure. Consider a numbering of its cells by integers $1,2,\ldots,n$. The cells of each translate carry the *induced numbering with respect to this translate*; namely, the cell corresponding to the ith cell of $\mathcal{F}$ has the number i.

1. Lemma 1. Mark a cell of the figure $\mathcal{F}$ and fix another cell A. Consider the following set of cells. A cell B belongs to this set if the translate of $\mathcal{F}$ with marked cell coinciding with B contains A. Then this set is a translate of $\overline{\mathcal{F}}$ and the cell A correspond to its marked cell.

Proof. Let V be the set of vectors connecting the marked cell of $\mathcal{F}$ with all its cells. Then both sets discussed coincide with the set of cells X such that the vector $\overrightarrow{XA}$ belongs to V.

1a). Let us introduce a bijection between transversal sets of $\overline{\mathcal{F}}$ (which are in one-to-one correspondence with those of $\mathcal{F}$) and $\mathcal{F}$-tilings. Given a set T, consider all translates of $\mathcal{F}$ with marked cells belonging to T. According to Lemma 1, each cell belongs to k translates, where k is the number of cells belonging to T in the corresponding translate of $\mathcal{F}$. It follows that we obtain a $\mathcal{F}$-tiling if and only if T is a transversal set for $\mathcal{F}$.

1b). Consider an arbitrary transversal set. Place in its cells $n-1$, and in other cells — (-1) (n is the area of the figure).

1c). Any tiling is uniquely determined by the set of marked cells. This set is a transversal set for $\overline{\mathcal{F}}$, and we can construct a barrier for $\overline{\mathcal{F}}$ (as in the previous solution). Thus it is enough to solve the next problem.

1d). Let d be the distance between the leftmost cell of the figure and its rightmost cell. It is clear that if we know the barrier on a segment of length d, then we can reconstruct this barrier. Since there are finitely many functions with a finite image on a finite set, there exist two segments with the same functions. The translation which identifies these segments identifies also the functions.

1e)$^\sharp$. According to the solution of **1a)**, it is enough to show that a set obtained by choosing a fixed cell in each translate of an $\mathcal{F}$-tiling is a transversal set for $\mathcal{F}$. Let us enumerate all the cells using the induced numberings with respect to the translates of our tiling (each cell is numbered exactly once). It is sufficient to prove that all numbers in each translate are distinct.

Suppose that there are two equal numbers (say, ones) in a translate $\mathcal{F}_0$. Let A and B be the cells containing these numbers. Suppose that they are numbered i and j w.r.t. $\mathcal{F}_0$. Consider the following translates of the tiling: $\mathcal{F}_1$ is the tiling with A being its first cell, $\mathcal{F}_2$ is the translate with B being its first cell. Clearly, jth cell of $\mathcal{F}_1$ and ith cell if $\mathcal{F}_2$ cover the same cell of space, which leads to the contradiction since $\mathcal{F}_1$ and $\mathcal{F}_2$ belong to the same tiling.

2a). Let us call the number of the leftmost cell of the translate *the number of the translate*. For the translate with number 0 the sum is equal to 0 by the definition. For the kth translate the sum is obtained from the sum for the translate with number 0 by multiplying it by ε^k and hence equals 0.

Consider the case of plane. Firstly, given a figure, define a polynomial as follows. Fix enumeration of cells with pairs of integers (coordinates) and form a polynomial

$$P_{\mathcal{F}}(x,y) = \sum_{i,j:\ (i,j)\text{th cell is contained in } \mathcal{F}} x^i y^j .$$

If two figures are translates of each other, the polynomials are obtained one from another by multplication by $x^k y^\ell$. Sometimes we do not distinguish between such polynomials.

Consider a pair of non-zero complex numbers (x_0,y_0) s.t. $P_{\mathcal{F}}(x_0,y_0)=0$ (notice that this condition is stable under multiplying the polynomial by monomials). Place in the (i,j)th cell the number $x_0^i y_0^j$. Clearly, we have obtained a barrier.

2b). Let $\mathcal{F}$ be an n-tile. Consider the set consisting of all the leftmost cells of translates belonging to our n-tiling. Let us call this set $\mathcal{G}$ a "characteristic set" of our n-tiling. The condition "given set of translates forms a tiling" can be rewritten as follows: "for any $0\leqslant m\leqslant n-1$ there exists a unique pair (k,l) s.t. $k+l=m$ and kth and lth cells belong to $\mathcal{F}$ and $\mathcal{G}$ respectively; for other m there is no such pair." One can easily rewrite it using polynomials: $P_{\mathcal{F}}(x)P_{\mathcal{G}}(x)=1+x+\ldots+x^{n-1}$.

It follows that $P_{\mathcal{F}}(x)$ divides $1+x+\ldots+x^{n-1}=\prod\limits_{d|n,\,d>1} F_d(x)$. All these factors are irreducible, and there are $d(n)$ of them, so we are done.

Remark. It is easy to see that we have also proved the following statement: any n-tile is symmetric w.r.t. a point (since all products of the polynomials $F_d(x)$ are symmetric) and for each n-tile there exists exactly one tiling.

3. Lemma 2. Let $\varepsilon_1,\ldots,\varepsilon_n$ be all roots of unity of degree n. Then $\varepsilon_1^k+\cdots+\varepsilon_n^k=0$ for all $1\leqslant k\leqslant n-1$.

Proof. Denote $\varepsilon_k=\cos\dfrac{2\pi k}{n}+i\sin\dfrac{2\pi k}{n}=\varepsilon_1^k$; then

$$\sum_{i=1}^{n}\varepsilon_i^k=\sum_{i=1}^{n}\varepsilon_k^i=\frac{\varepsilon_k^{n+1}-\varepsilon_k}{\varepsilon_k-1},$$

it vanishes since $\varepsilon_k^n=1$, while $\varepsilon_k\neq 1$.

3a). It is enough to prove that there exists a representation (1) for sequences $(b_k^{(i)})$, where

$$b_k^{(i)}=\begin{cases}1, & \text{if } k-i\,\vdots\, n,\\ 0, & \text{otherwise,}\end{cases}$$

since each n-periodic sequence is their linear combination. And for these sequences there exist following representations:

$$b_k^{(i)} = \frac{1}{n}\left(\varepsilon_1^{-i}\varepsilon_1^k + \dots + \varepsilon_n^{-i}\varepsilon_n^k\right)$$

(see Lemma 2).

3b). Let (a_k) be the barrier. Denote by $(a_k^{(i)})$ the function (a_k) shifted i units to the right: $a_k^{(i)} = a_{k-i}$; clearly, it is a barrier for $\mathcal{F}$. It follows that

$$c_k^{(i)} = a_k + a_k^{(1)}\varepsilon_i + a_k^{(2)}\varepsilon_i^2 + \dots + a_k^{(n-1)}\varepsilon_i^{n-1}$$

is also a barrier, since a linear combination of barriers is also a barrier. Substituting (a_k) from the previous problem, we obtain

$$c_k^{(i)} = \sum_{u=0}^{n-1} a_{k-u}\varepsilon_i^u = \sum_{u=0}^{n-1}\sum_{w=1}^{n} \alpha_w \varepsilon_w^{k-u}\varepsilon_i^u = \sum_{w=1}^{n} \alpha_w \varepsilon_w^k \sum_{u=0}^{n-1} \varepsilon_{i-w}^u.$$

Note that for $\varepsilon_j \neq 1$ we have $\sum_{u=0}^{n-1}\varepsilon_j^u = 0$, since ε_j is a root of the polynomial $x^n - 1 = (x-1)(1+\dots+x^{n-1})$; if $\varepsilon_j = 1$, we have $\sum_{u=0}^{n-1}\varepsilon_j^u = n$. It follows that $c_k^{(i)} = n\alpha_i\varepsilon_i^k$ is a barrier, so if $\alpha_i \neq 0$, then ε_i is a root of $P_{\mathcal{F}}$ and we are done.

3c). Consider a periodic tiling with period n. Find a representation (1) for its barrier. The first statement now follows from $\operatorname{lcm}(n_i)$-periodicity of any root function from that representation.

Suppose that the area of the figure equals m. Note that the numbers of all cells are pairwise distinct modulo n (otherwise some cells are covered with more than one figure). Consider all tiles s.t. the numbers of their leftmost cells are equal to the fixed number modulo n. Each of these figures covers m cells, their numbers having (fixed!) pairwise distinct remainders modulo n. It follows that these figures cover all the cells having these m remainders. Since all cells of each segment of length n can be split in such m-tuples, we are done.

3d). Consider the barrier corresponding to our transversal set. It has a representation (1) with ε_i being roots of unity which are also roots of $P_{\mathcal{F}}$. Then the numbers $\overline{\varepsilon}_i = \varepsilon_i^{-1}$ are roots of both $P_{\mathcal{F}}$ and $P_{\overline{\mathcal{F}}}$. It follows that this barrier is also a barrier for $\overline{\mathcal{F}}$, i. e. the transversal set is also a transversal set for $\overline{\mathcal{F}}$.