Algebraic tiling

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We will investigate tilings of the infinite (in both directions) cellular line (in problems 1 and 2 we will also consider tilings on the cellular plane, space etc.).

Definition. A *figure* is any finite set of cells (this set is not supposed to be connected!). It is clear that the area of a figure is a number of its cells (we assume that each cell has area 1). Any translation of a figure (on a vector with integer coordinates) is called a *translate* of this figure. Let $\mathcal{F}$ be a figure. A *$\mathcal{F}$-tiling* is a representation of a cellular line (plane, space etc.) as a disjoint union of translates of $\mathcal{F}$. A figure $\mathcal{F}$ is a *tile* if some $\mathcal{F}$-tiling exists.

A set of cells is a *transversal set* for a figure $\mathcal{F}$ if any translate of $\mathcal{F}$ intersects this set at exactly one cell.

We will also consider functions mapping the cellular line (plane, space) to $\mathbb{C}$, i. e. for each cell there is a complex number corresponding to it (further we will say that the number is *placed* into this cell). Such a function is called a *barrier function* for $\mathcal{F}$ if for any translate of $\mathcal{F}$ the sum of numbers in its cells is zero; a function is called *bounded* if there exists a number M such that absolute values of all numbers in cells do not exceed M.

Some hard problems are marked by a star. The problems which are not necessary for the further advantage in the main stream are marked by $\sharp$.

1. In this row of problems we consider tilings on arbitrary-dimensional cellular space.

1a). Prove that a figure is a tile if and only if there exists a transversal set for it. Moreover, the number of different tilings (maybe infinity) is equal to the number of different transversal sets for this figure.

1b). Prove that for any tile there exists a nontrivial bounded barrier function.

1c). Prove that any tiling on the cellular line is periodic.

1d). Consider a figure on the cellular line and suppose there exists its barrier function with a finite image. Prove that this function is periodic.

Let $\mathcal{F}$ be a figure. A figure symmetric to $\mathcal{F}$ with respect to a point (with coordinates $(a/2, b/2)$, $a,b \in \mathbb{Z}$) is denoted by $\overline{\mathcal{F}}$.

1e)$^\sharp$. Prove that any transversal set for $\mathcal{F}$ is also a transversal set for $\overline{\mathcal{F}}$.

1f)$^\sharp$. (A. Ya. Belov, 19 Tournament of Towns) Prove that the following conditions for a table-napkin are equivalent:

(i) If some its translates are placed on a table, then one can nail them up, any table-napkin being nailed by exactly one nail.

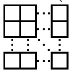
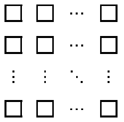
(ii) There exists a plane tiling consisting of translates of this table-napkin.

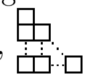
2. Let us enumerate the cells of the cellular line by integers in natural order. Let us assume that the number of the leftmost cell of $\mathcal{F}$ is 0. Associate to $\mathcal{F}$ a polynomial

$$P_{\mathcal{F}}(x) = \sum_{i:\ i\text{th cell is in } \mathcal{F}} x^i.$$

2a). Let ε be a root of the polynomial $P_{\mathcal{F}}$. Consider the function mapping the ith cell to ε^i. Prove that this function is a barrier for $\mathcal{F}$. Such a function will be referred to as a *root function* of $\mathcal{F}$. Try to formulate and prove a similar result for a figure on the cellular plane.

2b)$^\sharp$. A figure is a *n-tile* if some its translates form a tiling of the interval of length n. Prove that the number of different n-tiles does not exceed $2^{d(n)}$ (here, $d(n)$ is the number of divisors of n).

2c)$^\sharp$. We consider a cellular plane. Let the figure $\mathcal{F}$  consist of all the cells with coordinates (a,b): $0 \leqslant a,b \leqslant 2k$; let the figure $\mathcal{G}$  consist of all the cells with coordinates $(2a,2b)$: $0 \leqslant a,b \leqslant 2k$. Prove that any barrier for $\mathcal{F}$ is also a barrier for $\mathcal{G}$.

2d)$^\sharp$. Consider the functions mapping the cellular space to $\mathbb{Z}_2$ (residues modulo 2 with standard addition and multiplication), i. e. each cell has an element of $\mathbb{Z}_2$ placed in it. Find all integers $n>0$ satisfying the following property: any barrier for the "3-cell corner"  is also a barrier for the "cellular triangle"  with the length of its side equal to n.

All the following problems deal with tilings of the cellular line.

3. A complex number ε is called a *root of unity of degree n* if $\varepsilon^n = 1$. It is a *primitive root of degree n* if n is the least positive integer satisfying this property; in this situation n is called the *primitive degree* of ε. The set of all primitive roots of unity of degree n will be denoted by R_n.

3a). Let (a_k) be a periodic sequence with period n. Prove that it can be represented in the form

$$a_k = \alpha_1\varepsilon_1^k + \alpha_2\varepsilon_2^k + \ldots + \alpha_n\varepsilon_n^k, \tag{1}$$

where ε_i are all different roots of unity of degree n.

3b). Let $\mathcal{F}$ be a figure, (a_k) be its periodic barrier function. Prove that (a_k) can be represented in the form of 1 where ε_i are all roots of unity which are also the roots of the polynomial $P_{\mathcal{F}}$.

3c). Let n_i be all the primitive degrees of the roots of unity which are also the roots of $P_{\mathcal{F}}$. Prove that the period of each $\mathcal{F}$-tiling divides $\operatorname{lcm}(n_i)$ and is a multiple of the area of the tile $\mathcal{F}$.

3d). Solve the problem **1e)** (for the case of the cellular line) using the technique introduced.

4. (Generalization of S.L.Berlov's problem on the X Conference of the Tournament of Towns.) Here we will find a criterion for a figure of area p to be a tile (for a prime p). Any figure of area p will be further referred to as a *p-figure*.

Definition. A figure is *incompressible* if the greatest common divisor of the distances between its cells is 1 (informally, this mean that the cellular line cannot be "contracted", our tile lying in this "contraction"). Obviously, it is sufficient to investigate the tilings for incompressible tiles.

4a). Prove that for a given p-figure $\mathcal{F}$ there exists at most one positive integer k such that $P_{\mathcal{F}}$ has roots in R_{p^k}.

4b). If the polynomial of a p-figure has a root in R_{p^k} for $k>1$ then this figure is not incompressible.

4c)*. Let $\mathcal{F}$ be a tile of area n, where n is a multiple of a prime p. Find the number of such values of k that $P_{\mathcal{F}}$ has roots in R_{p^k}.

4d). Prove that an incompressible p-figure is a tile if and only if the numbers of all its cells have different residues modulo p.

Definition. A *linear combination* of some expressions $X_1,\ldots,X_n$ with rational (complex, integer, etc.) coefficients $a_1,\ldots,a_n$ is the expression $a_1X_1+\cdots+a_nX_n$.

5. The relation of this problem to the other ones does not seem obvious (but still is important!). We will investigate polynomials $P(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$ (with rational coefficients) having some roots in R_n. Such polynomials will be referred to as *n-root* polynomials.

Let n be a multiple of a prime p, $n=mp$. We call a *p-gonal polynomial* a polynomial in the form $x^i+x^{i+m}+\dots+x^{i+(p-1)m}$, $i<m$.

5a). Prove that a p-gonal polynomial is a n-root polynomial.

5b). Prove that any n-root polynomial is a linear combination of some p-gonal polynomials for some (maybe different) primes dividing n.

5c). Prove that if an n-root polynomial has integer coefficients, then it is a linear combination with integer coefficients of some p-gonal polynomials.

5d). Prove that if an n-root polynomial has only k nonzero coefficients then it is a linear combination of some p-gonal polynomials for different $p\leqslant k$.

5e)$^\sharp$. Suppose $\varepsilon_1,\dots,\varepsilon_5$ be roots of unity, $\varepsilon_1+\dots+\varepsilon_5=0$. Prove that these roots are either the vertices of a regular pentagon or the vertices of a regular triangle and two endpoints of a diameter on the complex circle $|z|=1$.

5f)$^\sharp$. (Proposed by K. A. Knop) Describe all the pairs of triangles with coinciding centroids having the vertices among the vertices of a given regular n-polygon.

6. These problems help to prove the following fact: for any incompressible p-tile $\mathcal{F}$ there exists only one $\mathcal{F}$-tiling (up to a translation).

By "different tilings" we mean the tilings that cannot be identified by a translation.

6a). Prove that for any composite $n>1$ there exists an incompressible tile $\mathcal{F}$ of area n with two different $\mathcal{F}$-tilings.

6b). Let $\mathcal{F}$ be a p-tile for which there exist two different $\mathcal{F}$-tilings. Prove that $P_{\mathcal{F}}$ has a root of order $n>p$, n being a multiple of p.

6c). In the case of the previous problem, prove that $\mathcal{F}$ is either p-times compressible or n/p-times compressible.

The following facts are considered known; you may use them without proof. Nevertheless, if you want you might try to prove them ...

Fact. a). Any polynomial with integer coefficients can be represented as a product of irreducible (over integers) polynomials; this representation is unique up to permutation of factors and multiplication (of some of) them by -1. Any polynomial with complex coefficients may be represented as a product of the form $a(x-x_1)\dots(x-x_n)$; this representation is unique up to permutation of factors.

b). The set of roots of the polynomial

$$F_{p^k}(x)=1+x^{p^{k-1}}+x^{2p^{k-1}}+\dots+x^{(p-1)p^{k-1}}$$

is R_{p^k}. This polynomial is irreducible (over $\mathbb{Z}$). Hence, if a polynomial $P(x)$ with integer coefficients has a root in R_{p^k} then all the elements of R_{p^k} are roots of $P(x)$.

c)*. The polynomial $F_n(x)=\prod_{\varepsilon\in R_n}(x-\varepsilon)$ is irreducible (over $\mathbb{Z}$).

Solutions

Let $\mathcal{F}$ be a figure. Consider a numbering of its cells by integers $1,2,\dots,n$. The cells of each translate carry the *induced numbering with respect to this translate*; namely, the cell corresponding to the ith cell of $\mathcal{F}$ has the number i.

1. Lemma 1. Mark a cell of the figure $\mathcal{F}$ and fix another cell A. Consider the following set of cells. Cell B belongs to this set if the translate of $\mathcal{F}$ with marked cell coinciding with B contains A. Then this set is a translate of $\overline{\mathcal{F}}$ and the cell A correspond to its marked cell.

Proof. Let V be the set of vectors connecting the marked cell of $\mathcal{F}$ with all its cells. Then both sets discussed coincide with the set of cells X such that the vector $\overrightarrow{XA}$ belongs to V.

1a). Let us introduce a bijection between transversal sets of $\overline{\mathcal{F}}$ (which are in one-to-one correspondence with those of $\mathcal{F}$) and $\mathcal{F}$-tilings. Given a set T, consider all translates of $\mathcal{F}$ with marked cells belonging to T. According to Lemma 1, each cell belongs to k translates, where k is the number of cells belonging to T in the corresponding translate of $\mathcal{F}$. This implies that we obtain a $\mathcal{F}$-tiling if and only if T is a transversal set for $\mathcal{F}$.

1b). Consider an arbitrary transversal set. Place $n-1$ in its cells, and in other cells of the cellular line (-1) (n is the area of the figure).

1c). Any tiling is uniquely determined by the set of marked cells. This set is a transversal set for $\overline{\mathcal{F}}$, and we can construct a barrier for $\overline{\mathcal{F}}$ (as in the previous solution). Thus it is enough to solve the next problem.

1d). Let d be the distance between the leftmost cell of the figure and its rightmost cell. It is clear that if we would know the barrier on a segment of length d, then we can reconstruct this barrier. Since there is only finite number of functions with a fixed finite image on a finite set, then there exist two segments with the same functions. The translation which identifies these segments also identifies the functions.

1e)$^\sharp$. According to the solution of **1a)**, it is enough to show that a set obtained by choosing a fixed cell in each translate of an $\mathcal{F}$-tiling is a transversal set for $\mathcal{F}$. Let us enumerate all the cells using the induced numberings with respect to the translates of our tiling (each cell is numbered exactly once). It is sufficient to prove that all numbers in each translate are distinct.

Suppose that there are two equal numbers (say, ones) in a translate $\mathcal{F}_0$. Let A and B be the cells containing these numbers. Suppose that they are numbered i and j w.r.t. $\mathcal{F}_0$. Consider the following translates of the tiling: $\mathcal{F}_1$ is the tiling with A being its first cell, $\mathcal{F}_2$ is the translate with B being its first cell. Clearly, jth cell of $\mathcal{F}_1$ and ith cell of $\mathcal{F}_2$ cover the same cell of space, which leads to the contradiction since $\mathcal{F}_1$ and $\mathcal{F}_2$ belong to the same tiling.

1f)$^\sharp$. To be frank, the authors didn't realize completely all the technical difficulties at the time. So, it is very pleasant that some participants have solved it, although maybe with loss of rigour.

Denote our napkin by $\mathcal{F}$. Let us mark some point on $\mathcal{F}$. Consider the set of the marked points of all $\mathcal{F}$-translates that can be nailed up with a particular nail. This set is a translate of the napkin $\overline{\mathcal{F}}$ with this nail at the marked point. Thus, we formulate condition (i) as follows:

(iii) Let us mark some points on the plane. Then there exists a set of $\overline{\mathcal{F}}$-translates such that each marked point is covered with exactly one translate.

Let us show that (iii) holds provided by that there exists $\overline{\mathcal{F}}$-tiling (i. e. (ii) $\Longrightarrow$ (iii)). Actually, let us shift our tiling so that no marked point lies on the boundary of any translate of the tiling. Obviously, in such case each point is covered with exactly one translate. Why can we move the tiling in such a way? Each points specify some "forbidden" translations which form the boundary of tiling translates. It is clear (really?) that a finite number of such sets cannot cover the whole plane.

The idea of the converse implication (iii) $\Longrightarrow$ (ii) is also quite simple. Let us mark a "very dense" set of points on the large part of plane. Then a set of translates covering them should form a set "close to a tiling" over this part. Thus, it seems to be a correct statement that a tiling also exists. The rest part of this solution is devoted to the formalization of this idea.

Compactness Lemma. Assume that k infinite bounded sequences $(a_n^{(i)})$ $(i=1,\dots,k)$ are given. Then there exists a sequence of indexes $j_1,j_2,\dots$ such that for each i the sequence $a_{j_1}^{(i)},a_{j_2}^{(i)},a_{j_3}^{(i)},\dots$ converges.

Proof. Induction on k. The base is well-known. Suppose that a sequence of indices (p_n) is found such that for any $i=1,2,\dots,k-1$ the sequence $(a_{p_n}^{(i)})$ converges. Applying the base to the (bounded!) sequence (b_n), where $b_n=a_{p_n}^{(k)}$ we can find a subsequence $q_1,q_2,\dots$, such that the sequence (b_{q_r}) is convergent. Then the sequence $j_n=p_{q_n}$ is a desired one.

Let us choose such a unit of length that a napkin may be covered by a 1×1 square; let S be its area. Consider the square $2i\times 2i$ with the center at the origin, and construct the set V_i of marked points consisting of all points of this square with the coordinates in the form $\left(\frac{p}{2^k};\frac{q}{2^k}\right)$ with $p,q\in\mathbb{Z}$. Let W_i be a placement of $\overline{\mathcal{F}}$-translates corresponding to V_i. Clearly, the placement W_i is also a correct placement for V_ℓ (provided that $\ell<i$) since $V_\ell\subset V_i$. Therefore, we can replace W_ℓ by W_i. We will use this idea in the following situation. Let a subsequence W_{i_1}, W_{i_2}, … $(i_1<i_2<\cdots)$ be chosen. Then all the other W_i's may be replaced by some terms of this subsequence. We will proceed in a similar way for all W_j with $j\geqslant k$ when we perform the kth step of the following process.

On the kth step we fix the square $2k\times 2k$ with the center at the origin. For any W_i this square intersects not more then $4(k+1)^2/S$ translates contained in W_i (actually, all these translates are arranged in the square $2(k+1)\times 2(k+1)$). Therefore, we can choose a subsequence for which the number of such translates is constant (applying the previous remark, we may assume that this number is constant for all but a finite number of W_i's). Let $(x_i^{(1)},y_i^{(1)}),\dots,(x_i^{(d)},y_i^{(d)})$ be the coordinates of marked points of these translates. By applying Compactness Lemma we can choose a subsequence of indices such that all the sequences (x_{i_j}), (y_{i_j}) converge. Again, by the terms of the subsequence chosen we replace all the W_j's with the indices $j\geqslant k$ being not in this subsequence. The kth step is complete.

Note that when kth step is over, any point of V_k either is covered by the interior of exactly one translate or lies on the boundary (but not inside!) of some translates of the limit placement of the translates.

Recall that we did not change the set W_j for $j\leqslant k$ after the kth step. Hence there exists a sequence of placements that can be called as the *result* of our process. Moreover, for any k a sequence of sets of translates that intersect the $2k\times 2k$ square converges (this was the main goal of performing our process). Thus, we can construct the limit placement of translates over the whole plane (take the limit placement on the 2×2-square, spread it onto the limit placement on the 4×4-square, etc.)

Let us prove that the placement constructed is a tiling (huh!). Actually, if there is a point in the square $2k\times 2k$ (for some k) which is not covered with every translate, then there is a little circle centered at this point which does not intersect with any translate. Then there exists a point contained in some V_ℓ (for $\ell>k$) inside this circle. But this point should be covered by some limit translate. Conradiction. A point (with some circle around it) cannot be covered twice by the limit placement since this wasn't the case in all but a finite number of W_k's.

2a). Let us call as the *number of the translate* the number in the leftmost cell of the translate. The sum for the translate with number 0 is equal to 0 by the definition. For the kth translate the sum is obtained from the sum for the translate with number 0 by multiplying it by ε^k; hence it equals 0.

Consider the case of plane. Firstly, given a figure, define a polynomial as follows. Fix enumeration of cells with pairs of integers (coordinates) and form a polynomial

$$P_{\mathcal{F}}(x,y)=\sum_{i,j:\ (i,j)\text{th cell is contained in }\mathcal{F}} x^i y^j.$$

If two figures are translates of each other, the polynomials are obtained one from another by multplication by $x^k y^\ell$. Sometimes we do not distinguish between such polynomials.

Consider a pair of non-zero complex numbers (x_0,y_0) s.t. $P_{\mathcal{F}}(x_0,y_0)=0$ (notice that this condition is stable under multiplying the polynomial by monomials). Place in the (i,j)th cell the number $x_0^i y_0^j$. Clearly, we obtain a barrier.

2b). Let $\mathcal{F}$ be an n-tile. Consider the set consisting of all the leftmost cells of translates belonging to our n-tiling. Let us call this set $\mathcal{G}$ a "characteristic set" of our n-tiling. The condition "given set of translates forms a tiling" can be rewritten as follows: "for any $0 \leqslant m \leqslant n-1$ there exists a unique pair (k,l) s.t. $k+l=m$ and kth and lth cells belong to $\mathcal{F}$ and $\mathcal{G}$ respectively; for other m there is no such pair." One can easily rewrite it using polynomials: $P_{\mathcal{F}}(x)P_{\mathcal{G}}(x)=1+x+\ldots+x^{n-1}$.

It follows that $P_{\mathcal{F}}(x)$ divides $1+x+\ldots+x^{n-1}=\prod\limits_{d|n,\, d>1} F_d(x)$. All these factors are irreducible, and there are $d(n)$ of them, so we are done.

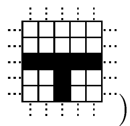
Remark. It is easy to see that we have also proved the following statement: any n-tile is symmetric w.r.t. a point (since all products of the polynomials $F_d(x)$ are symmetric) and for each n-tile there exists exactly one tiling.

2c). In fact, for an arbitrary polynomial it is possible to define a barrier. If a polynomial corresponds to a figure (i. e., all its coefficients are 0 and 1), a barrier for this polynomial is simply a barrier for the corresponding figure. In general case the definition is left to the reader.

Note that the barrier for a polynomial is also a barrier for all its multiples. Indeed, for any position of figure PQ summation over its cells is a linear combination of sums over some positions of figure P; each of these sums is equal to 0.

We have: $P_{\mathcal{F}}=(1+x+\cdots+x^{2k})(1+y+\cdots+y^{2k})$ $P_{\mathcal{G}}=(1+x^2+x^4+\cdots+x^{4k})(1+y^2+y^4+\cdots+y^{4k})$. It remains to notice that $P_{\mathcal{G}}=P_{\mathcal{F}}(1-x+x^2-\cdots+x^{2k})(1-y+y^2-\cdots+y^{2k})$.

2d). **Answer.** For $n=2^k-2$ $(k>1)$.

Let us show that it happens if and only if the "triangle polynomial" $Q_n(x,y)=\sum_{i+j\leqslant n-1} x^i y^j$ is a multiple of the "corner polynomial" $Q_2(x,y)=1+x+y$ (we consider polynomials with coefficients in $\mathbb{Z}_2$). If it is a multiple, then the statement is clear (it is similar to the previous solution). Suppose that it is not a multiple. Obviously, we can decrease the y-degree by subtracting a multiple of $1+x+y$. It follows that $Q_n(x,y)=Q_2(x,y)R(x,y)+T(x)$ with $T(x)\neq 0$. It remains to notice that there exists a barrier for a 3-cell corner with arbitrary "initial data" (numbers in the given horizontal line). Indeed, it is uniquely determined by its values in the following cells: ); it is easy to find the placement of numbers in the horizontal line s.t. this function would not be a barrier for $T(x)$. Contradiction.

Question. For which n $Q_n(x)$ is a multiple of $Q_2(x)$? If n is odd then it is not valid (consider $x=1$, $y=0$). Suppose that n is even. Then $Q_n(x,y)=Q_{n/2}(x^2,y^2)Q_2(x,y)+xyQ_{n/2-1}(x^2,y^2)$. Since we have $(a+b)^2=a^2+b^2+2ab=a^2+b^2$ over $\mathbb{Z}_2$, then $Q_{n/2-1}(x^2,y^2)=Q_{n/2-1}(x,y)^2$. It follows that if it holds for n then it is true for $n/2-1$. Since the least n for which it is true is 2, it is quite easy to prove that the above answer is valid.

3. Lemma 2. Let $\varepsilon_1,\ldots,\varepsilon_n$ be the set of all roots of unity of degree n. Then $\varepsilon_1^k+\cdots+\varepsilon_n^k=0$ for all $1\leqslant k\leqslant n-1$.

Proof. Denote $\varepsilon_k=\cos\dfrac{2\pi k}{n}+i\sin\dfrac{2\pi k}{n}=\varepsilon_1^k$; then

$$\sum_{i=1}^{n}\varepsilon_i^k=\sum_{i=1}^{n}\varepsilon_k^i=\frac{\varepsilon_k^{n+1}-\varepsilon_k}{\varepsilon_k-1},$$

it vanishes since $\varepsilon_k^n=1$, while $\varepsilon_k\neq 1$.

3a). It is enough to prove that there exists a representation (1) for sequences $(b_k^{(i)})$, where

$$b_k^{(i)} = \begin{cases} 1, & \text{if } (k-i) \vdots n, \\ 0, & \text{otherwise,} \end{cases}$$

since each n-periodic sequence is their linear combination. Note that for these sequences there exist following representations:

$$b_k^{(i)} = \frac{1}{n}\left(\varepsilon_1^{-i}\varepsilon_1^k + \dots + \varepsilon_n^{-i}\varepsilon_n^k\right)$$

(see Lemma 2).

3b). Let (a_k) be a barrier. Denote by $(a_k^{(i)})$ the function (a_k) shifted i units to the right: $a_k^{(i)} = a_{k-i}$; clearly, it is a barrier for $\mathcal{F}$. It follows that

$$c_k^{(i)} = a_k + a_k^{(1)}\varepsilon_i + a_k^{(2)}\varepsilon_i^2 + \dots + a_k^{(n-1)}\varepsilon_i^{n-1}$$

is also a barrier, since a linear combination of barriers is also a barrier. Substituting (a_k) from the previous problem, we obtain

$$c_k^{(i)} = \sum_{u=0}^{n-1} a_{k-u}\varepsilon_i^u = \sum_{u=0}^{n-1}\sum_{w=1}^{n} \alpha_w \varepsilon_w^{k-u}\varepsilon_i^u = \sum_{w=1}^{n} \alpha_w \varepsilon_w^k \sum_{u=0}^{n-1} \varepsilon_{i-w}^u.$$

Note that for $\varepsilon_j \neq 1$ we have $\sum_{u=0}^{n-1}\varepsilon_j^u = 0$, since ε_j is a root of the polynomial $x^n - 1 = (x-1)(1+\dots+x^{n-1})$; if $\varepsilon_j = 1$, we have $\sum_{u=0}^{n-1}\varepsilon_j^u = n$. It follows that $c_k^{(i)} = n\alpha_i\varepsilon_i^k$ is a barrier, so if $\alpha_i \neq 0$, then ε_i is a root of $P_{\mathcal{F}}$ and we are done.

3c). Consider a periodic tiling with period n. Find a representation (1) for its barrier. The first statement follows from $\operatorname{lcm}(n_i)$-periodicity of any root function used in that representation.

Suppose that the area of the figure equals m. Note that the numbers of all cells are pairwise distinct modulo n (otherwise some cells are covered with more than one figure). Consider all tiles s.t. the numbers of their leftmost cells are equal to the fixed number modulo n. Each of these figures covers m cells with their numbers having (fixed!) pairwise distinct residues modulo n. It follows that these figures cover all the cells having these m residues. Since all cells of each segment of length n can be split in such m-tuples, we are done.

3d). Consider the barrier corresponding to our transversal set. It has a representation (1) with ε_i being roots of unity which are also roots of $P_{\mathcal{F}}$. Then the numbers $\overline{\varepsilon}_i = \varepsilon_i^{-1}$ are roots of both $P_{\mathcal{F}}$ and $P_{\overline{\mathcal{F}}}$. It follows that this barrier is also a barrier for $\overline{\mathcal{F}}$, i. e. the transversal set is also a transversal set for $\overline{\mathcal{F}}$.

4a). Let F be our p-figure. Its area equals $P_{\mathcal{F}}(1) = p$. Assume $P(x)$ has roots in $R_{p^{k_1}}$ and in $R_{p^{k_2}}$. According to **b)**, a polynomial $P_{\mathcal{F}}(x)$ is divisible by $F_{p^k}(x)$ provided that it has a root in R_{p^k}. Polynomials $F_{p^{k_1}}(x)$ and $F_{p^{k_2}}(x)$ have no common roots if $k_1 \neq k_2$, hence they are coprime. Since $P_{\mathcal{F}}(x)$ has roots in both sets $R_{p^{k_1}}$ and $R_{p^{k_2}}$, then it is divisible by $F_{p^{k_1}}(x)F_{p^{k_2}}(x)$, and the quotient has integer coefficients according to **a)**. However, $P_{\mathcal{F}}(1) = p$ and $F_{p^{k_1}}(1)F_{p^{k_2}}(1) = p^2$, therefore the quotient has value of $1/p$ at 1. This contradicts to the fact that the coefficients of the quotient are integers.

4b). In similar way, we get $P_{\mathcal{F}}(x) = F_{p^k}(x)Q(x)$, where $Q(x)$ is a polynomial with integer coefficients. By grouping powers with the fixed residues modulo p^{k-1}, one can represent the polynomial $Q(x)$ in the form

$$Q(x) = Q_0(x^{p^{k-1}}) + xQ_1(x^{p^{k-1}}) + \dots + x^{p^{k-1}-1}Q_{p^{k-1}-1}(x^{p^{k-1}}).$$

Then we have

$$P_{\mathcal{F}}(x) = \sum_{i=0}^{p^{k-1}-1} P_i(x), \qquad \text{where} \quad P_i(x) = x^i Q_i(x^{p^{k-1}})F_{p^k}(x),$$

where the powers of x with nonzero coefficients are distinct in different polynomials $P_i(x)$; therefore all the coefficients of polynomials $P_i(x)$ are nonnegative. Moreover, since $F_{p^k}(1)=p$ we obtain that $P_i(1)$ is divisible by p. The relation $P_i\mathcal{F}(1)=p$ implies that only one of these terms is nonzero. Since the zeroth term of $P_{\mathcal{F}}(x)$ equals unity, we have

$$P_{\mathcal{F}}(x)=P_0(x)=Q_0(x^{p^{k-1}})F_{p^k}(x)=Q_0(x^{p^{k-1}})F_p(x^{p^{k-1}})$$

and the figure is p^{k-1}-times compressible.

4c)*. **Answer.** $\mathrm{ord}_p n$, that is, the maximal j such that $p^j\,|\,n$

Using the ideas similar to those of **4a)**, one can easily check that the sought number in question does not exceed $\mathrm{ord}_p n$. Let the tiling have a period m. Consider interval of length m. Let us mark all the cells corresponding to the leftmost cells of the tiling translates. This set will be referred to as the "characteristic set" $\mathcal{G}$ of our tiling (by shifting the interval along the tiling we may assume that zeroth cell belongs to $\mathcal{G}$).

The solution of **3c)** implies that if one chooses only the figures placed on these cells then exactly one cell for each residue modulo m is left. In terms of polynomials, this criterion means that $P_{\mathcal{F}}(x)P_{\mathcal{G}}(x)\equiv 1+x+\dots+x^{m-1} \pmod{x^m-1}$. Note that this statement is symmetric with respect to $\mathcal{F}$ and $\mathcal{G}$. Therefore we may assume $\mathcal{G}$ being a tile, and $\mathcal{F}$ being a "characteristic set" of the tiling. Then, the number of k's for which $P_{\mathcal{G}}(x)$ is divisible by F_{p^k} doesn't exceeds $\mathrm{ord}_p\ell$, where $\ell=P_{\mathcal{G}}(1)$ is the area of $\mathcal{G}$ $(n\ell=m)$. Note, however, that the polynomials $1+x+\dots+x^{m-1}$ and x^m-1 are divisible by each F_{p^k} for $1\leqslant k\leqslant \mathrm{ord}_p m$. Therefore, the polynomial $P_{\mathcal{F}}(x)P_{\mathcal{G}}(x)$ is divisible by $\mathrm{ord}_p m=\mathrm{ord}_p n+\mathrm{ord}_p\ell$ such polynomials. So, both unequalities mentioned in fact are equalities, implying that the sought number for $P_{\mathcal{F}}(x)$ is $\mathrm{ord}_p n$, while that of $P_{\mathcal{G}}(x)$ is $\mathrm{ord}_p\ell$.

4d). Obviously, if all cells have different residues, then all shifts on vectors that are multiples of p compose a tiling.

Let us prove the converse statement. From above it follows that the polynomial $P_{\mathcal{F}}(x)$ is divisible by $F_p(x)$. We are left to prove that $P_{\mathcal{F}}(x)\equiv 1+x+\dots+x^{p-1} \pmod{x^p-1}$. On the contrary, there is at least one residue (suppose i) which is not present in $P_{\mathcal{F}}(x)$ among the powers of x. So, $x^{p-i-1}P_{\mathcal{F}}(x)\equiv Q(x) \pmod{x^p-1}$, where $\deg Q(x)\leqslant p-2$, $Q(x)\neq 0$ (check the value at 1!). However, any primitive root of unity of degree p is a common root of polynomials $P_{\mathcal{F}}(x)$ and x^p-1; hence the polynomail $Q(x)$ has at least $p-1$ roots. Contradiction.

5a). Let ε be a primitive root of unity of degree n. Then $\varepsilon^i+\varepsilon^{i+m}+\dots+\varepsilon^{i+(p-1)m}=\varepsilon^i(1+\varepsilon^m+\dots+\varepsilon^{(p-1)m})$. In brackets we get all the roots of unity of degree p, their sum being zero.

5b). It is well-known that the gcd of two polynomials can be represented as their linear combination (with the polynomial coefficients). Using the induction one can show that this statement holds also for any finite set of polynomials. Hence any polynomial that is a multiple of this gcd may also be represented in the form

$$P(x)=\sum_{p|n}Q_p(x)(1+x^{n/p}+\dots+x^{(p-1)n/p}).$$

If one changes any $Q_p(x)$ by its residue modulo $x^{n/p}-1$, then the polynomial $P(x)$ changes by its residue modulo x^n-1, i.e. it remains unchanged, and one obtains the desired representation.

Since any n-root polynomial is a multiple of $F_n(x)$, we are left to prove that the gcd of all p-gonal polynomials is $F_n(x)$.

The set of roots of the p-gonal polynomial $1+x^m+\dots+x^{(p-1)m}=\dfrac{x^n-1}{x^{n/p}-1}$ coincides with the set of all roots of unity of degree n which are not the roots of degree n/p; all these roots are simple. Thus the roots of the gcd desired are those roots of unity of degree n which are not the roots of degree n/p for any $p\,|\,n$, i.e. all the primitive roots of degree n, QED.

Now we introduce the language which is useful for investigation of n-root polynomials.

While exploring n-rooted polynomials, it is convenient to consider the residues of our polynomials modulo x^n-1. One can interprete this idea combinatorially as follows. A polynomial is associated with the n-tuple of numbers arranged on a circle (on the ith position the sum of all coefficients with their numbers having the residues i modulo n is placed). As a particular case, the polynomial $x^i(1+x^{n/p}+x^{2n/p}+\dots+x^{(p-1)n/p})$ is associated with the set of p 1's placed at the vertices of the regular p-gon (this is the reason to call such polynomial "p-gonal"). The operations of addition and multiplication are introduced in a natural way so that a sum and a product of two polynomials is associated with the sum and the product of their images (details are left to the reader). Hereinafter, we denote by p and q some prime divisors of n.

Fix naturals i and d and consider the set of all positions in the form $i+kd$ on the n-circle. Such a set is called a d-contraction (actually, we may assume that $d\,|\,n$). One can easily check that there are exactly d different d-contractions. A *trace* of the polynomial on the d-contraction is a tuple of all its coefficients placed in this contraction. Obviously, the sum of traces equals the trace of the sum; the product of the traces equals the trace of the product iff the contraction contains 0th position. Hence, the structure of the n/d-circle may be introduced on the d-contraction, and we will sometimes associate an n/d-circle with the d-contraction of the n-circle.

From what mentioned above it follows that all the coefficients of a q-gonal poloynomial are contained in some p-contraction provided that $q\neq p$; hence all the traces of q-gonal polynomial on such contraction are n-root. As a particular case, denoting by n_0 the product of all the prime divisors of n, one can obtain that each q-gonal polynomial is contained in some n/n_0-contraction. Thus, a trace of an n-root polynomial on such contraction is also n-root. Hence, in some cases it is sufficient to investigate n-root polynomials only for the case of n being square-free.

Since $(x^{n/p}-1)(1+x^{n/p}+\dots+x^{(p-1)n/p})\equiv 0 \pmod{x^n-1}$, one can obtain that the following statement holds.

Lemma (on the rotation of the n-circle). Assume $P(x)$ is an n-root polynomial, $n=p^i d$. Then the trace of the polynomial $(x^{n/p}-1)P(x)$ on any p^i-contraction is also n-root.

5c). From the above text we have that any n-root polynomial can be represented as a linear combination of p-gonal ones. It follows that the multiplication of any n-root polynomial by a polynomial in the form of $Q(x)=(1-x^{k_1n/p_1})(1-x^{k_2n/p_2})\cdots(1-x^{k_ln/p_\ell})$ turns our polynomial to zero (here, $p_1,\dots,p_\ell$ are all prime factors of n). Actually, one can represent the initial polynomial as a sum of some p-gons and then separate it into "components" – the sum of all p_1-polygons, the sum of all p_2-polygons, etc. The sum of all p_i-polygons vanishes under the multiplication by $(1-x^{k_in/p_i})$, thus it vanishes also under multiplication by $Q(x)$. Hence the whole sum of p-gons becomes zero too.

Assume that n is square-free (i. e. $n=n_0$). It is well-known (cf. the polynomial reasoning in the previous solution) that in this case any K can be uniquely represented in form

$$K\equiv k_1n/p_1+k_2n/p_2+\dots+k_\ell n/p_\ell \pmod{n}, \qquad 0\leqslant k_i<p_i. \tag{$*$}$$

We say that the coefficient is *marked* if in the representation of its number in the form $(*)$ at least one of k_i's is zero.

We claim that if all marked coefficients of an n-root polynomial are equal to zero then it is zero itself. Assume that the (nonmarked!) coefficient a_t is nonzero with its number t being represented in the form $(*)$ by means of the set $k_1,\dots,k_\ell$. Multiplying our n-root polynomial by $(1-x^{-k_1(n/p_1) \pmod{n}})(1-x^{-k_2(n/p_2) \pmod{n}})\cdots(1-x^{-k_l(n/p_\ell) \pmod{n}})$, we get zero. Let us consider the zeroth coefficient of the polynomial obtained. This coefficient is a linear combination of some coefficients of the initial polynomial, all of these coefficients except a_t being marked. We have that the marked coefficients are equal to zero, and so is the linear combination. However, the coefficient over a_t in this combination is nonzero, hence $a_t=0$.

Thus, for given n-root polynomial it is sufficient to find an integer linear combination of p-gons such that the subtraction of this combination from the initial polynomial makes all its marked coefficients zero. We will make them zero by induction on number of nonzero k_j's in the representation $(*)$ of the coefficient's number.

Substracting an appropriate p-polygon one can vanish the zeroth coefficient. Assume that all the coefficients having less than t nonzero k_i's in their representation are zero. Consider any coefficient with the number having exactly t different nonzero k_j's. Take any i such that $k_i=0$. We subtract the p_i-gonal polynomial containing this coefficient so that our coefficient becomes 0. One can easily see that the representations (of the form $(*)$) of the other coefficients' numbers in our p_i-gon differs from the given one only in ith component; hence, our procedure leaves all the obtained zeroes unchanged, and we are done.

5d). Expand our polynomial F into a sum of its traces $x^iP_i(x^p)$ on p-contractions. Suppose that every such trace is n-root. Then we can view every trace as an n/p-root polynomial. Since each such polynomial has no more nonzero coefficients than the initial polynomial has, we may proceed by induction on n. Actually, if the polynomial $P_i(x)$ can be represented as a linear combination of p-gons on the n/p-circle then such representation is also valid for the polynomial $P_i(x^p)$ on the n-circle.

On the contrary, assume that one of the traces is not n-root. Then the corresponding trace of the sum of all the p-gons in the representation of $P(x)$ is not n-root (since all q-gons for $q\neq p$ are contained in one p-contraction and thus the sum of their traces form a n-root polynomial). Thus the traces of $P(x)$ on *all* the p-contractions are not n-rooted (since the traces of p-gons may be mapped one into the other by an appropriate rotation); hence they are not zero. Therefore, each trace contains some nonzero coefficient of $P(x)$. This implies that $p\leqslant k$.

Remark. Let a given n-root polynomial be represented as a linear combination of q-gons with all q's not exceeding p, where p is the maximal prime factor of n. A p-gon is called *necessary* for the polynomial $P(x)$ if a trace of $P(x)$ on some pn/n_0-contraction is not n-root (recall that n_0 is the maximal square-free divisor of n; the number $u=pn/n_0$ is chosen so that any q-gon except of p-gons is contained in some u-contraction). Then the following arguments show that one can represent $P(x)$ as a linear combination of q-gons for some different $q<p$ iff a p-gon is not necessary for $P(x)$. Moreover, Lemma 3 can be sharpened as follows.

Lemma 4 (on existence of p-gon). Assume $P(x)$ is an n-root polynomial with nonnegative integer coefficients, $P(1)=p$. If a p-gon is necessary for $P(x)$ then $P(x)$ is p-gonal itself.

Proof. It follows from the above solution that $P(x)$ is in some n/n_0-contraction, hence we may assume that $n=n_0$, i. e. n is square-free. Then every trace of p-gons on a p-contraction is not zero, and they can be obtained one from the other by a rotation. So, the conditions of the Lemma imply that the sum of coefficients of any trace is 1, i. e. there is exactly one coefficient in each p-contraction. Multiply our polynomial by $x^{n/p}-1$. By Lemma 3, every trace of the polynomial obtained on a p-contraction is n-root. Denote the trace on rth contracton by $P_r(x)$.

Any polynomial $P_r(x)$ has one coefficient of 1 and one of -1 (these coefficients may annihilate each other). Suppose they did not annihilate, i. e. $P_r(x)$ has the form of x^a-x^b, where $a\equiv b \pmod p$ (we may assume wlog that $a>b$). However, the polynomial $x^a-x^b=(x^{a-b}-1)x^b$ can not be n-root since for any primitive root ε of degree n we have $P_r(\varepsilon)=(\varepsilon^{a-b}-1)\varepsilon^b\neq 0$, given that $0<a-b<n$. Contradiction.

Hence, any $P_r(x)$ is zero. This implies that a polynomial $P(x)$ contains the term $x^{k+n/p \pmod n}$ if it contains the term x^k. Then it already contains the terms $1, x^{n/p}, \ldots, x^{(p-1)n/p}$ (we may assume wlog that $P(x)$ contains the term 1). Since $P(1)=p$, there are no other terms in $P(x)$, QED.

5e)$^\sharp$. We may assume wlog that one of the roots is 1. Let n be the lcm of the primitive degrees of the roots given, ε be a primitive root of degree n. Then one can rewrite the sum in question as a n-root polynomial with nonnegative integer coefficients: $P(\varepsilon)=0$, $P(1)=5$. We need to prove that $P(x)$ is either 5-gonal or the sum of 3- and 2-gonal polynomials. From the text above it follows that all $p>5$ are not necessary for $P(x)$.

Suppose 5-gon is necessary. By Lemma 4 we obtain that $P(x)$ is a 5-gon itself. In the converse case, we claim that it is a desired sum. Actually, we will prove the stronger statement than the one we need (this statement will also be useful below): *Suppose a polynomial $P(x)$ with nonnegative integer coefficients may be represented as a linear combination of 2- and 3-gons; then it can be represented in such form with nonnegative coefficients.* Obviously, the initial statement is the consequence of this one. We remark that any trace of $P(x)$ on any $n/6$-contraction is n-root; hence it is sufficient to prove our statement for a polynomial with its coefficients being placed on the 6-circle.

Note that the sum of the coefficients over an arbitrary 3-gon and an arbitrary 2-gon is nonnegative (since there is a coefficient of $P(x)$ coinciding with this sum). Moreover, we may change the sum of three different 2-ons by the sum of two different 3-gons (and vice versa). Let us use this procedure to increase all the negatve coefficients over 2-(3-)gons; then (by the previous remark) no negative coefficient over a 3-(2-)gon will appear. Hence, when this procedure is over, our representation has the desired form.

5f)$^\sharp$. Firstly, we formulate our problem in more convenient terms. Associate the vertices of our regular n-gon with the roots of unity of degree n in a natural way. Then one can rewrite the condition for the triples (a_1,a_2,a_3), (b_1,b_2,b_3) of vertices of the desired triangles as follows: $a_1+a_2+a_3=b_1+b_2+b_3$. Consider the numbers $c_i=-b_i$ (the corresponding points are symmetric to b_i with respect to the origin). Then c_i are roots of unity of degree $\operatorname{lcm}(2,n)$, and our condition is rewritten in the form $\sum a_i+\sum c_i=0$; conversely, if a_i,c_i are six roots satisfying this property, then the points a_i, b_i are close to compose the desired pair of triangles. It is left to clarify whether b_i are the roots of degree n (but not only $2n$). For the convenience, let us exchange n by $\operatorname{lcm}(2,n)$ for a while.

So, we need to describe the 6-tuples of roots with their sum being zero. We will argue using the scheme of the previous solution. Only 2-, 3- and 5-gons may be necessary. Suppose 5-gons are necessary. Then there exist (exactly!) five non-n-root traces of $P(x)$ on $5n/n_0$-contractions. Obviously, one (actually, at least four) of them has only one nonzero coefficient. Let us subtract from $P(x)$ the 5-gon intersecting this coefficient. Then this trace (and thus all the traces) become n-root.

Consider our five $5n/n_0$-contractions (that is, those on which our polynomial had induced non-n-root traces). It was mentioned that at least four of these traces had contained exactly one nonzero coefficients. Then each of them become either 0 or a polynomial of the form x^a-x^b; we mentioned in the proof of Lemma 4 that the latter case is impossible. Thus, the difference has only two coefficients of 1 and one of -1. The arguments similar to above shows that this is the difference between 3-gon and 2-gon with on common vertice. This situation will be referred to as $5+3-2$; this is the only case when 5-gon is necessary.

Assume on the contrary that they are not necessary. Then the statement from the previous solution implies that only two situations may occur: two 3-gons ("the situation $3+3$") and three 2-gons ("the situation $2+2+2$"). Hence, only three situations ($5+3-2$, $3+3$, and $2+2+2$) may appear. We are left to clarify which pairs of triangles are associated with these situations (one situation may be associated with more than one pair since a 6-tuple may be split into triples in different ways!).

(i) $2+2+2$. If a triangle has exactly one root from each pair then the resulting **triangles coincide**. Otherwise, the symmetry being made we obtain **a pair of right triangles with the common right vertice** (this means that n should be even).

(ii) $3+3$. Either all three vertices of one 3-gon are the vertices of a triangle (this is the case of **two regular triangles** ($n\,\vdots\,3$)); or a triangle contains two vertices of the first 3-gon and one vertice of another 3-gon. This is the first essentially nontrivial case: **each triangle has one side subtending the arc of $4\pi/3$ on the circumcircle; the third vertice of one triangle is the center of this arc (note that this arc is greater than half-circle) and vice versa** (this also needs $n\,\vdots\,3$).

(iii) $5+3-2$. Recall that this means the following combination: the vertices of a regular 5-gon and a regular 3-gon, one vertice of 5-gon being opposite to that of 3-gon; this pair is eliminated. Obviously, our configuration is contained among the verices of a regular 30-gon; moreover, we shall see that it is possible only for $n\,\vdots\,30$. Let us enumerate the vertices of our 30-gon in a natural order; then our

6-tuple may have the numbers of 6,12,18,24,5,25 (the vertices 0 and 15 are eliminated). Now we can write down all the possible variants:
1) **(5,6,12)**, **(3,9,10)**; 2) **(5,6,18)**, **(9,10,27)**; 3) **(5,12,24)**, **(3,10,21)**; 4) **(5,18,24)**, **(10,21,27)**; 5) **(5,6,25)**, **(3,9,27)**; 6) **(5,12,25)**, **(3,9,21)**; 7) **(5,6,24)**, **(3,10,27)**.

Remark. These methods are also useful for the investigation of the sums of the greater number of roots. However, the length of arguing will grow up since the number of cases grows significantly.

6a). Suppose $n=dk$, $d,k>1$. We construct the following figure:

$$\underbrace{\square\square\cdots\square}_{d}\underbrace{\qquad\qquad}_{2d}\underbrace{\square\square\cdots\square}_{d}\underbrace{\qquad\qquad}_{2d}\cdots\underbrace{\qquad\qquad}_{2d}\underbrace{\square\square\cdots\square}_{d}$$

(there are k d-blocks in this figure). According to **1a)**, it is sufficient to find two transversal sets which are not equivalent wrt a translation. These sets consist of all cells with the numbers having residues (i) $0,d,2d$; (ii) $0,2d,4d$ modulo n.

Remark. Obviosly, this figure may be "compressed" in another way: the cellular line may be split into d-blocks in such way that each block either is contained in a figure or does not intersect it. The authors know how to construct a figure which is not compressible in both ways (but still has more than one tiling).

6b). For a p-tile there exists a unique p-periodic tiling (a *standard* tiling). So, if a p-figure allows two different tilings, one of them does not have a period p. Assume the contrary: the primitive degree of any root of $\mathcal{F}_P$ divisible by p is p. According to **4c)** there exists such root. Denote by (a_k) the barrier constructed by the means of the nonstandard tiling (cf. **1b)**). Note that this function has only two different values.

Consider the representation of (a_k) in the form of (1). Denote the sum of all the terms, corresponding to the roots of degree p by c_k, and the sum of all other terms by b_k; $a_k=b_k+c_k$. Note that the period of (b_k) is not divisible by p because the period of any its term is not divisible by p.

Note that the sequence (c_k) is not constant since the period of (a_k) is divisible by p. Then the period of (c_k) is p. But (b_k) is also not a constant because the period of (a_k) is greater than p. Denote the period of (b_k) by d. Hence there exist such k and ℓ that $b_k\neq b_0$, $c_\ell\neq c_0$. Now, by Chinese remainder theorem, there exist such r_{k0}, $r_{0\ell}$, $r_{k\ell}$, that $r_{0\ell}\equiv 0 \pmod d$, $r_{k0}\equiv r_{k\ell}\equiv k \pmod d$, $r_{k0}\equiv 0 \pmod p$, $r_{0\ell}\equiv r_{k\ell}\equiv \ell \pmod p$. Obviously, there exist at least three distinct numbers among $a_0=b_0+c_0$, $a_{r_{k0}}=b_k+c_0$, $a_{r_{0\ell}}=b_0+c_\ell$, $a_{r_{k\ell}}=b_k+c_\ell$; but the sequence (a_k) is two-valued. Contradiction.

6c). Denote by $P(x)$ the residue of $P_{\mathcal{F}}(x)$ modulo x^n-1 (recall that $n>p$ is the degree of some root of unity which is also a root of $P_{\mathcal{F}}(x)$, $n\,\vdots\,p$), $P(1)=p$; then $P(x)$ is n-root. It is sufficient to show that $P(x)$ is p-gonal (clearly, in such case all the monomials in $P_{\mathcal{F}}(x)$ are multiples of n/p). By **4d)**, the degrees of nonzero monomials of $P(x)$ have all possible residues modulo n, and for every residue there exists exactly one such monomial (here we use the fact that $\mathcal{F}$ is p times incompressible). Hence, even on p-contraction the trace of $P(x)$ is not n-root, which implies that a p-gon is necessary for $P(x)$. Then Lemma 4 completes the proof.

7. (ii) $\Longleftrightarrow$ (iii) according to the solutions of **1a)** and **1e)**: each set of equally coloured cells is a transversal set, and a colouring can be easily contructed (cf. **1e)**). It is also obvious that (ii) $\Longrightarrow$ (i). Hence, it is sufficient to prove (i) $\Longrightarrow$ (iii).

Clearly, while dealing with the polynomials of figure and a "characteristic set" of a tiling, we work with the residues *modulo* x^p-1 *and* y^p-1 (that is, each power of x or y greater than $p-1$ can be lowered using one of these polynomials; thus, these residues have the form of $\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}a_{ij}x^iy^j$). So, we can speak of the *value of a residue at the point* (ε,δ) iff this point is a common root of x^p-1 and y^p-1 (i. e., $\varepsilon^p=\delta^p=1$); otherwise this value can be changed while adding one of these polynomials. Hence, all ε,δ are assumed to be the roots of 1 of degree p in the text below.

Since there exists a "k-times tiling," we can construct a polynomial $P_{\mathcal{G}}(x)$ (with integer coefficients) such that $P_{\mathcal{F}}(x,y)P_{\mathcal{G}}(x,y)=kF_p(x)F_p(y)$; on the other hand, $P_{\mathcal{F}}(x,y)F_p(x)F_p(y)=pF_p(x)F_p(y)$. Note that the polynomials $pP_{\mathcal{G}}(x,y)$ and $kF_p(x)F_p(y)$ are distinct (the former has some zero coefficients!), hence we obtain the nonzero polynomial $Q(x) = pP_{\mathcal{G}}(x,y) - kF_p(x)F_p(y)$ such that $P_{\mathcal{F}}(x,y)Q(x,y) = 0$; moreover, the degrees of Q over x and y do not exceed $p-1$.

We claim that the polynomial $P_{\mathcal{F}}(x,y)$ has a root (ε,δ) (such that $\varepsilon^p = \delta^p = 1$). At first, let us show that this implies the desired statement. The barrier obtained from this root has p different values. Moreover, as mentioned in the solution of **4d)**, the sum of p such values equals 0 iff they are distinct. Hence, the cells which contain the coinciding values of the barrier being coloured in one colour, then the desired colouring is constructed.

We are left to prove our claim. Assume the contrary. Then $Q(x,y)$ has *all* these roots (recall that $P_{\mathcal{F}}Q = 0$). Represent the polynomial $Q(x,y)$ in the form $Q(x,y) = Q_0(x) + Q_1(x)y + \cdots + Q_{p-1}(x)y^{p-1}$ and substitute any $x = \varepsilon$; we obtain a polynomial in y having at least p roots (that are, all the δ's). However, the degree of this polynomial is $\leqslant p-1$, so this polynomial equals zero and all its coefficients are zeroes. This means that any $Q_i(x)$ has p roots (all ε's), while their degrees do not exceed $p-1$. Hence they are zeroes, therefore $Q(x,y) = 0$. Contradiction completes the proof.

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