Comparative Geometry of the Triangle and the Tetrahedron

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The additional result in task 7 part 1, that was found by participants of the conference.

During the Conference M. Isaev (Barnaul) and V. Filimonov (Ekaterinburg) discovered the following condition. Namely, the segments are concurrent if and only if the touching point of the insphere and one of the faces coincides with the Torricelli point of this face. Let us prove that all the touching points are Torricelli points in this case.

So let A', B', C', D' be the touching points and $\angle AD'B = \angle BD'C = \angle CD'A = \frac{2\pi}{3}$. Since AD' and AC' are tangent segments with a common endpoint, $AD' = AC'$. The same is true for BD' and BC'. It follows that the triangles ABD' and ABC' are congruent and $\angle AC'B = \angle AD'B = \frac{2\pi}{3}$. The same arguments show that $\angle AB'C = \angle BA'C = \frac{2\pi}{3}$. Moreover, $\angle AB'D = \angle AC'D$, $\angle BA'D = \angle BC'D$, $\angle CA'D = \angle CB'D$, $\angle AB'D + \angle CB'D = \angle AC'D + \angle BC'D = \angle BA'D + \angle CA'D = \frac{4\pi}{3}$. It follows that all these angles are equal (to $\frac{4\pi}{3}$).

Since D' and C' are the Torricelli points of the corresponding faces, the lines DC' and CD' are bisectors of the angles $\angle AC'B$ and $\angle AD'B$. Therefore the lines DC' and CD' intersect AB at a point X s.t. $\frac{AX}{BX} = \frac{AD'}{BD'} = \frac{AC'}{BC'}$. It follows that the points C, D, C', D' belong to the plane CDX and so the segments CC' are DD' concurrent. The same is true for each pair of segments, so all these segments are concurrent (since they are not coplanar).

The other implication is left to the reader.