Comparative Geometry of the Triangle and the Tetrahedron

A. Zaslavsky

Solutions

I. Introductory problems

1. Let C', D' be the centroids of faces ABD, ABC, P be the midpoint of AB. Since C', D' belong to the plane CDP, the segments CC' and DD' intersect at point M. Clearly, $CD' : D'P = DC' : C'P = 2 : 1$, and it follows that $DM : MD' = CM : MC' = 3 : 1$.

2. Let D' be the circumcenter of ABC. The points of the line passing through D' and perpendicular to the plane ABC are equidistant from the points A, B, C. The plane passing through the midpoint of AD and perpendicular to AD intersect this line in point O which is the circumcenter of tetrahedron. Moreover, we proved that the perpendiculars to faces of tetrahedron passing from its circumcenters intersect at O.

3. a) Firstly, we shall construct the insphere touching all faces. Given trihedral angle $DABC$, consider the bisectors of its dihedral angles $\angle(DAB, DAC)$ and $\angle(DBA, DBC)$. The points of the first plane are equidistant from the planes DAB and DAC, the points of second — from DAB and DBC, so all points of their intersection line are equidistant from all three planes. The bisector of the dihedral angle $\angle(ABC, ABD)$ intersects this line in the incenter. Similarly one can prove that there exist 4 exspheres touching one of the faces and the extensions of three other faces. But sometimes there exists a sphere touching the extensions of two faces beyond their common edge and the extension of two remaining faces beyond the endpoints of this edge. If this sphere exists for the edge CD, its radius is equal to $\frac{3V}{S_{ABC}+S_{ABD}-S_{ACD}-S_{BCD}}$. So if $S_{ABC}+S_{ABD} > S_{ACD}+S_{BCD}$ there exists a sphere touching the extensions of two faces beyond CD, if $S_{ABC}+S_{ABD} < S_{ACD}+S_{BCD}$ — a sphere touching the extensions of two faces beyond AB, if $S_{ABC}+S_{ABD} = S_{ACD}+S_{BCD}$ none of these two spheres exists. Thus it can be from 5 to 8 spheres touching all planes of the faces of a tetrahedron.

b) Let the sphere touch all edges of tetrahedron. Since all tangents to sphere from the same point are equal, $AB + CD = AC + BD = AD + BC$. This condition is also sufficient and such tetrahedrons are called *skeleton*. There are several equivalent "skeletonity conditions". For example, the incircles of any two faces touch or the sums of opposite dihedral angles are equal. Similarly can be proved that the sphere touching three edges of the one face and the extensions of three other edges exists if and only if the differences of opposite edges are pairwise equal.

4. Let the altitudes CC' and DD' intersect. The plane containing them intersectis AB in the point P. Line CP is the projection of CC' onto ABC, so by the "three perpendiculars theorem" it is perpendicular to AB. As AB is also perpendicular to DD', it is perpendicular to plane CPD, and hence to CD. Thus if all altitudes of tetrahedron intersect at one point, its opposite edges are perpendicular. This condition is also sufficient and respective tetrahedrons are called *orthocentric*. There are several equivalent conditions. For example, the sum of squares of opposite edges are equal or the foundation of one altitude is the orthocenter of respective face.

5. Since the area of face is equal to the the sum of areas of remaining faces' projections, we have a system

$$\begin{cases} S_{ABC} = S_{BCD}\cos\angle(ABC,BCD) + S_{CDA}\cos\angle(ABC,CDA) + S_{DAB}\cos\angle(ABC,DAB) \\ S_{BCD} = S_{CDA}\cos\angle(BCD,CDA) + S_{DAB}\cos\angle(BCD,DAB) + S_{ABC}\cos\angle(BCD,ABC) \\ S_{CDA} = S_{DAB}\cos\angle(CDA,DAB) + S_{ABC}\cos\angle(CDA,ABC) + S_{BCD}\cos\angle(CDA,BCD) \\ S_{DAB} = S_{ABC}\cos\angle(DAB,ABC) + S_{BCD}\cos\angle(DAB,BCD) + S_{CDA}\cos\angle(DAB,CDA) \end{cases}$$

Multiplying first equation by S_{ABC}, second — by S_{BCD}, third — by S_{CDA}, fourth — by S_{DAB}, and subtracting second equation from the sum of three other equations, we obtain the statement of the theorem.

6. We start with the **sines theorem for a trihedral angle**: the ratio of sines of the dihedral angles is equal to the ratio of sines of the opposite flate angles. Indeed, let H be the projection of the point C from the edge of trihedral angle $OABC$ onto the plane OAB. then $CH = OC\sin AOC\sin\angle(OAB,OAC) = OC\sin BOC\sin\angle(OBC,OAB)$. Now by this theorem and the sines theorem for a triangle we obtain

$$\frac{AB\cdot CD}{AC\cdot BD} = \frac{\sin ACB\sin DBC}{\sin ABC\sin BCD} = \frac{\sin\angle(ABC,ABD)\sin\angle(ACD,BCD)}{\sin\angle(ABC,ACD)\sin\angle(ABD,BCD)}.$$

Before the discussion of the remaining problems we prove the following important fact.

Lemma. Let C', D' belong to the faces ABD and ABC. The lines CC' and DD' intersect if and only if

$$\frac{S_{AD'C}}{S_{BD'C}} = \frac{S_{AC'D}}{S_{BC'D}}.$$

Proof. Since CC' and DD' intersect C, D, C', D' belong to the same plane intersecting AB at point P. So

$$\frac{AP}{BP} = \frac{S_{ACP}}{S_{BCP}} = \frac{S_{AD'P}}{S_{BD'P}} = \frac{S_{AD'C}}{S_{BD'C}},$$

q.e.d.

Recall that the numbers $\alpha = \frac{S_{BCD'}}{S_{ABC}}$, $\beta = \frac{S_{ACD'}}{S_{ABC}}$, $\gamma = \frac{S_{ABD'}}{S_{ABC}}$ are called *barycentric coordinates* of D' with respect to the triangle ABC.

7. Let D' is a touching point of face ABC wuth the insphere, X, Y — the projections of D' onto AC and BC respectively, r — radius of the insphere. Then $D'X = r/\tan\frac{1}{2}\angle(ABC,ACD)$, $D'Y = r/\tan\frac{1}{2}\angle(ABC,BCD)$. Using analogous equality for the face ABD, Lemma and the sines theorem, we obtain that CC' and DD' intersect if and only if

$$\cos\frac{1}{2}\angle(ABC,ACD)\cos\frac{1}{2}\angle(ABD,BCD) = \cos\frac{1}{2}\angle(ABD,ACD)\cos\frac{1}{2}\angle(ABC,BCD).$$

So these four segments are concurrent if and only if the products of cosines of half opposite dihedral angles are equal.

8. Similarly to previous problem we obtain that these segments intersect if and only if the products of sines of half opposite dihedral angles are equal.

9. Let M_1, O_1, H_1 are the centroid, the circumcenter and the orthocenter of triangle ABC. Since the tetrahedron $ABCD$ is orthocentric, DH_1 is its altitude. Consider the plane $M_1O_1H_1D$. Clearly, it contains the centroid M, the orthocenter H and the circumcenter O of the tetrahedron. Similarly, this points belong to the analogous planes constructed for other faces. It follows that O, M, H belong to the line which is the intersection of these planes. It is also easy to prove that M is the midpoint of OH.

10. This follows from Lemma and the obvious fact: products of barycentric coordinates of isotomically conjugated points are equal to each other.