Comparative Geometry
of the Triangle and the Tetrahedron

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Most of these problems can be formulated as follows. Given a property which holds for each triangle, find an analogue of this property for tetrahedrons and determine the class of tetrahedrons for which this analogue holds. (It is clear that one can propose more problems like that, and this list is not complete. . .)

I. Introductory problems

Although many of these problems are well-known it is useful to solve them "for training".

1. The three medians of a triangle are concurrent and their common point ("centroid") divide each of them in the ratio 2:1. — The four segments connecting the vertices of a tetrahedron with the centroids of the opposite faces are concurrent and their common point divideeach if them in the ratio 3:1.

2. For any triangle, there exists a circumscribed circle. — For any tetrahedron, there exists a circumscribed sphere.

3. For any triangle, there exist an inscribed circle ("incircle") and three exscribed circles ("excircles"). — a) How many spheres which touching all planes containing tetrahedron's faces does there exist? b) Does there exist a sphere touching all edges of the tetrahedron (for any tetrahedron) ?

4. The altitudes of triangle are concurrent. — For which tetrahedrons the altitudes are concurrent?

5. Prove **the cosines theorem for the tetrahedron**: for a tetrahedron $ABCD$,

$$S^2_{BCD} = S^2_{ABC} + S^2_{ABD} + S^2_{ACD} - 2S_{ABC}S_{ABD}\cos\angle(ABC, ABD) - \\ - 2S_{ABD}S_{ACD}\cos\angle(ABD, ACD) - 2S_{ACD}S_{ABC}\cos\angle(ACD, ABC)$$

(S denotes the area, $\angle$ — angle between the planes).

6. Prove **the sines theorem for the tetrahedron**: for a tetrahedron $ABCD$,

$$\frac{AB \cdot CD}{AC \cdot BD} = \frac{\sin\angle(ABC, ABD)\sin\angle(ACD, BCD)}{\sin\angle(ABC, ACD)\sin\angle(ABD, BCD)}.$$

7. The three segments connecting the vertices of a triangle and the touching points of opposite sides with the incircle are concurrent (their common point is called Gergonne's point). — For which tetrahedrons the four segments connecting the vertices and the touching points of opposite faces with the insphere are concurrent?

8. The three segments connecting the vertices of a triangle and the touching points of opposite sides with the corresponding exscribed circles are concurrent (their common point is called Nagel's point). — For which tetrahedrons the four segments connecting the vertices and the touching points of opposite faces with the corresponding exscribed spheres are concurrent?

9. Centroid, orthocenter and circumcenter of a triangle are collinear (Euler's line). Does analogous property hold for an orthocentric tetrahedron?

10. (Isotomic conjugation) Let ABC be a triangle. For a point P, denote by A', B', C' the intersection points of the lines AP, BP, CP and the opposite sides of the triangle. Then the lines connecting the vertices A, B, C with the points symmetric to A', B', C' with respect to the midpoints of the corresponding sides are concurrent. — Formulate an analogous property for a tetrahedron.

II. Main problems

11. (Isogonal conjugation) Let ABC be a triangle. For a point P, denote by l_1, l_2, l_3 the lines symmetric to AP, BP, CP with respect to the bisectors of the corresponding angles of the triangle. Then the lines l_1, l_2, l_3 are concurrent. — a) Formulate an analogous propety for a tetrahedron. b) Let A', B', C', D' be the projections of a point P onto the planes BCD, DCA, DAB, ABC respectively. Prove that the perpendiculars dropped from A, B, C, D onto respective faces of the tetrahedron $A'B'C'D'$ are concurrent.

12. The touching points of a side of a triangle with the incircle and the respective excircle are symmetric with respect to the midpoint of that side. — a) Prove that the touching points of a face of a tetrahedron with the insphere and the respective exsphere are isogonally conjugated. b) Which point is isogonally conjugated to touching point of the lateral face with the exsphere touching the foundation? c) Prove that the touching point of lateral face with the exsphere touching the foundation is in circumcenter of this face if and only if the sum of areas of this face and the foundation is equal to the sum of areas of remaining lateral faces.

13. The centroid of a triangle is isogonally conjugated to the Lemuan's point (which minimizes the sum of squares of distances to the sides of the triangle). Does analogous property hold for a tetrahedron?

14. For which classes of tetrahedrons the segments connecting the vertices with
a) orthocenters of the opposite faces;
b) Gergonne's points of the opposite faces;
c) Nagel's points of the opposite faces;
d) incenters of the opposite faces;
e) Lemuan's points of the opposite faces
are concurrent?

15. Prove for pp. a-e) of the previous problem that if two of the indicated segments are concurrent then the remaining two are also concurrent.

16. Prove the statement of previous problem for the segments connecting the vertices and the touching points of opposite faces with the insphere and the respective exspheres.

III. Extra problems

Most of this problems have not been solved yet.

17. The centroid, the incenter and the Nagel's point of a triangle are collinear. Does this property hold for some classes of tetrahedrons?

18. The centroid and the circumcenter of a triangle are isogonally conjugated. Does analogous property hold for an orthocentric tetrahedron?

19. The Gergonne's and Nagel's points of a triangle are isogonally conjugated to centers of homotethic transformations mapping the incircle to the circumcircle. Does analogous property hold for some classes of tetrahedrons?

20. The Gergonne's point of a triangle is the Lemuan's point of the triangle formed by the touching points of sides and the incircle ("intouched triangle"). Does analogous property hold for some classes of tetrahedrons?

21. The line connecting the incenter and the circumcenter of a triangle pass through the centroid of its intouched triangle. Does analogous property hold for a tetrahedron?

22. Solve problem 14 for a) the circumcenters; b) Torricelli's points (Torricelli's point of ABC is the common point of lines AA', BB', CC', where A', B', C' are the vertices of equilateral triangles ABC', BCA', CAB', disposed out of ABC); c) Apollonius points (points that are isogonally conjugated to the Torricelli's points; the projections of the Apollonius point onto the sides form an equilateral triangle and the distances from this point to the vertices are inversely proportional to the lengths of respective sides).

23. Does the statement of problem 15 hold for points from the previous problem?

24. Define the analogues of Torricelli's and Apollonius points (for some classes of tetrahedrons).