Comparative Geometry of the Triangle and the Tetrahedron

A. Zaslavsky

Solutions

II. Main problems

11. a) Firstly, we make the following observation. Consider a trihedral angle $OABC$ and a ray OP. The reflection of plane OCP with respect to the bisector of the dihedral angle $\angle(OAC, OBC)$ and two analogous planes for OA, OB contain a common line OP'. It can be easy proved using the sines theorem for a trihedral angle. We shall call line OP' conjugated to OP with respect to the trihedral angle $OABC$.

Now we prove the following statement. The lines conjugated to AP, BP, CP, DP with respect to the corresponding trihedral angles are concurrent. In fact if the ratio of distances from P to planes BCD, CDA, DAB, ABC is $a : b : c : d$, then each of conjugated lines pass through point P', for which this ratio is $\frac{1}{a} : \frac{1}{b} : \frac{1}{c} : \frac{1}{d}$. We shall call P' *isogonally conjugated* to P.

b) Let the plane $A'B'C'$ cut off from $ABCD$ the tetrahedron $DUVW$. then the distances from P to ABD, ACD, BCD are inversely proportional to the cosines of dihedral angles adjacent to the foundation of tetrahedron $PUVW$ and the distances of any point of perpendicular from D to $A'B'C'$ are proportional to these cosines. So all four perpendiculars are passing through the point P' which is isogonally conjugated to P.

12. a) Under solution of problems 7, 8 we proved that the ratio of the distances from the touching point of any face with insphere to sides of this face is equal to the ratio of cotangents of half dihedral angles adjacent to the respective edges of this face, and the ratio of the distances from the touching point of the respective exsphere is equal to the ratio of tangents of these angles. So the touching points are isogonally conjugated.

b) Similarly to (a) we obtain that the touching point of the face ABC with the exsphere touching the face ABD and the extensions of remaining faces is isogonally conjugated to the touching point of the face ABC with the sphere touching the extensions faces ABC and ABD beyond AB or the extensions of faces ACD and BCD beyond CD (depending on existence of these spheres).

c) This immediately follows from (b) and the well known fact: the point does not have an isogonally conjugated point if and only if it belongs to the circumcircle (the respective lines are parallel).

13. Since the distances from the centroid to the faces of tetrahedron are inversely proportional to the areas of these faces, the distances from isogonally conjugated point are proportional to the areas. We shall prove that this point minimize the sum of squares. Let S_a, S_b, S_c, S_d be the areas of faces, a, b, c, d — the distances from any point to faces, V — the volume of tetrahedron. Then $aS_a + bS_b + cS_c + dS_d = 3V$ and

$$a^2 + b^2 + c^2 + d^2 = a^2 + b^2 + c^2 + \left(\frac{3V - aS_a - bS_b - cS_c}{S_d}\right)^2$$

As polynomial of degree 2 with respect to c, this expression reaches its minimum if $c = \frac{-aS_aS_c-bS_bS_c+3VS_c}{S_c^2+S_d^2}$. This is equivalent to $c(S_c^2+S_d^2) = 3VS_c - aS_aS_c - bS_bS_c$, and it follows that $cS_d^2 = S_c(3V - aS_a - bS_b - cS_c) = dS_cS_d$, and $\frac{c}{S_c} = \frac{d}{S_d}$.

From analogous equalities for a and b we obtain $a : b : c : d = S_a : S_b : S_c : S_d$ q.e.d.

14. a) From solution of problem 4 it follows that such a tetrahedron is orthocentric.

b) Barycentric coordinates of the Gergonne's point are inverse proportional to $p-a$, $p-b$, $p-c$, where a, b, c are the lengths of sides of a triangle, $p = \frac{1}{2}(a+b+c)$. Using Lemma, we see that such a tetrahedron is skeleton.

c) Barycentric coordinates of the Nagel's point are proportional to $p-a$, $p-b$, $p-c$. Using Lemma, we see that such a tetrahedron is skeleton.

d) Barycentric coordinates of the incenter are proportional to a, b, c. Using Lemma, we see that the products of opposite edges of such a tetrahedron are equal. Such tetrahedrons are called *incentric.*

e) Barycentric coordinates of the Lemuan's point are proportional to a^2, b^2, c^2. Using Lemma, we see that such a tetrahedron is incentric.

15–16. Using Lemma, it is easy to prove that the intersection conditions for segments from A and B and from C and D coincide.

III. Supplementary problems (some hints)

18. It is not true (see problem 11b).

24. If all angles of triangle are less than $\frac{2\pi}{3}$ then the Torricelli's point T minimizes the sum of distances to the vertices and $\angle ATB = \angle BTC = \angle CTA (= 120^\circ)$. Similarly if there exists a point T inside the tetrahedron such that $\angle ATB = \angle CTD$, $\angle ATD = \angle BTC$, $\angle ATC = \angle BTD$, then T minimizes the sum of distances to the vertices. In this case the projection of point T' isogonally conjugated to T onto the faces form an equilateral tetrahedron (the tetrahedron is called *equilateral* if its faces are pairwise equal triangles. There are many conditions equivalent to equilaterality, for example opposite edges are equal, or opposite dihedral angles are equal, or two of three points — the centroid, the circumcenter and the incenter — coincide). There is a conjecture that the condition of existence of point T is as follows: the sum of dihedral angles adjacent to any vertex is less than 2π (equivalent condition: the sum of cosines of flat angles is greater than -1).

The additional result in task 7 part 1, that was found by participants of the conference.

During the Conference M. Isaev (Barnaul) and V. Filimonov (Ekaterinburg) discovered the following condition. Namely, the segments are concurrent if and only if the touching point of the insphere and one of the faces coincides with the Torricelli point of this face. Let us prove that all the touching points are Torricelli points in this case.

So let A', B', C', D' be the touching points and $\angle AD'B = \angle BD'C = \angle CD'A = \frac{2\pi}{3}$. Since AD' and AC' are tangent segments with a common endpoint, $AD' = AC'$. The same is true for BD' and BC'. It follows that the triangles ABD' and ABC' are congruent and $\angle AC'B = \angle AD'B = \frac{2\pi}{3}$. The same arguments show that $\angle AB'C = \angle BA'C = \frac{2\pi}{3}$. Moreover, $\angle AB'D = \angle AC'D$, $\angle BA'D = \angle BC'D$, $\angle CA'D = \angle CB'D$, $\angle AB'D + \angle CB'D = \angle AC'D + \angle BC'D = \angle BA'D + \angle CA'D = \frac{4\pi}{3}$. It follows that all these angles are equal (to $\frac{4\pi}{3}$).

Since D' and C' are the Torricelli points of the corresponding faces, the lines DC' and CD' are bisectors of the angles $\angle AC'B$ and $\angle AD'B$. Therefore the lines DC' and CD' intersect AB at a point X s.t. $\frac{AX}{BX} = \frac{AD'}{BD'} = \frac{AC'}{BC'}$. It follows that the points C, D, C', D' belong to the plane CDX and so the segments CC' are DD' concurrent. The same is true for each pair of segments, so all these segments are concurrent (since they are not coplanar).

The other implication is left to the reader.