Triangulations

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Problems

In these problems, we assume that all the sets of points on the plane (for example, vertices of polygons) are in a generic position.

1 *Triangulations of polygons*

1.1. Prove that an arbitrary polygon can be cut into triangles by means of diagonals that do not intersect in inner points.

There is a very popular number sequence in combinatorics — the Catalan numbers c_n $(n \geqslant 0)$. It is defined by a recurrence relation

$$c_{n+1} = c_0 c_n + c_1 c_{n-1} + \ldots + c_n c_0 \quad (n \geqslant 0)$$

and "initial value" $c_0 = 1$. An exact formula for Catalan numbers is also known: $c_n = \frac{1}{n+1} C_{2n}^n$.

It is not too difficult to check that a convex n-gon can be cut into triangles by c_{n-2} ways.

One of the other combinatorial realizations of Catalan numbers is a number of planted strictly binary trees (i.e., each vertex has either 2 or 0 sons (in the latter case this vertex is by definition a leaf) with $n+1$ leaves) (see fig. 1).

1.2. Find a bijective proof of the fact that the number of planted strictly binary trees with $n+1$ leaves is the Catalan number c_n.

1.3. In how many ways one can cut an arbitrary convex n-gon into triangles such that each triangle contains an outer side of the n-gon?

A triangulation of a polygon is defined as its partition into triangles by means of nonintersecting diagonals. Sometimes we also mean by a triangulation just the set of these diagonals.

Let $n > 3$ be a natural number. We say that triangulations T_1 and T_2 of n-gons P_1 and P_2 are *compatible* if one can enumerate the vertices of these n-gons by numbers $1, 2, \ldots, n$ along the border in such a way that a diagonal connecting two vertices of P_1 belongs to the triangulation T_1 if and only if a diagonal connecting the vertices of P_2 with the same numbers belongs to the triangulation T_2.

It is evident that for arbitrary two n-gons a compatible triangulation may not exist (see fig. 2)

Let several points be marked inside of an n-gon. A partition of this n-gon into triangles by means of nonintersecting segments with endpoints in the vertices of the n-gon or in the marked points is called *a triangulation with additional vertices.* The additional vertices are also called Steiner points.

We say that two n-gons P_1 and P_2 have *a compatible triangulation with k additional vertices* if we can mark k points in each n-gon, enumerate the vertices of the n-gons by numbers $1, 2, \ldots, n$ along the border, enumerate the additional vertices by numbers $n+1, n+2, \ldots, n+k$, and after that triangulate these n-gons in a compatible manner, which means that for every segment in the triangulation of P_1 there exists a segment connecting vertices with the same numbers in the triangulation of P_2.

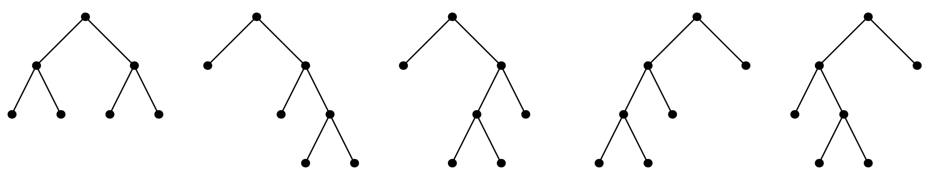

Рис. 1: $c_3 = 5$

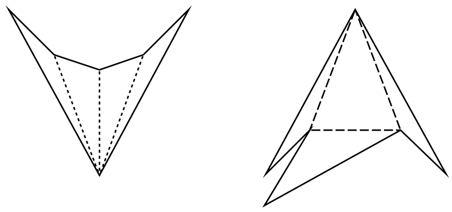

Рис. 2: Two polygons that can not be triangulated compatibly

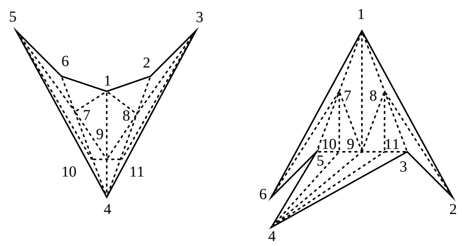

Рис. 3: A compatible triangulation

1.4. Every triangulation of a convex n-gon necessarily contains a diagonal which cuts off a triangle. We will call such a triangle (or the vertex that was cut off) *an ear* of the triangulation.

а) Prove that every triangulation of a (not necessarily convex) polygon has at least 2 ears.

б) Prove that the number of triangulations of a convex n-gon that have exactly k ears is equal to

$$\frac{n}{k}2^{n-2k}C_{n-4}^{2k-4}c_{k-2}\,.$$

в) Prove that approximately a quarter of vertices of a polygon are in average ears of a triangulation. In other words, “an average” triangulation has about $n/4$ ears.

1.5. Consider a triangulation of a right n-gon ($n \geqslant 5$) (possibly with additional vertices) such that all its triangles are

а) acute;

b) obtuse.

What is the least possible number of triangles which this traingulation can have?

1.6. Prove that all the triangulations of a convex n-gon with k Steiner points have equal number of triangulations and equal number of edges.

1.7. Prove that for any two n-gons there exists a compatible triangulation with additional vertices (the number of additional points may depend on the polygons).

It occurs that there exists a universal triangulation with additional vertices.

1.8. а) Prove that for any $n > 3$ there exists k such that every n-gon can be triangulated with k additional vertices and all the triangulations obtained in this way will be pairwise compatible.

б) Prove that there exists a universal triangulation in a stronger sense: for any n there exists k such that every triangulation of every n-gon can be split by adding k additional vertices in such a manner that all these triangulations will be pairwise compatible.

1.9. Let $\kappa = \kappa(n)$ be the minimal k such that it is possible to triangulate with k additional vertices all the n-gons so that all triangulations obtained will be pairwise compatible. Prove that there exist constants c_1, c_2, $c_3 > 0$, such that

а) $\kappa \leqslant c_1 n^2$;

б) $c_2 n^2 \leqslant \kappa$, if we assume that the order of vertices of every polygon is fixed;

в) $c_3 n^2 \leqslant \kappa$, if we assume that the order of vertices of every polygon is not fixed.

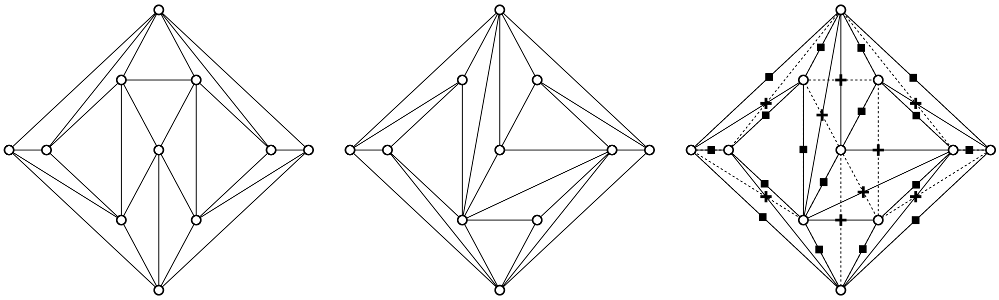

Рис. 4: For the problem 2.5.a: identical matched edges are marked by boxes, intersection points of two crossing matched edges are marked by crosses.

2 *Triangulations of arbitrary sets*

Let us remind that we consider the point sets in the plane in a generic position.

A triangulation of an arbitrary point set is by definition a partition of a convex hull of these points into triangles by segments that have endpoints in these points only.

We say that two sets S_1 and S_2 of n points each have *a compatible triangulation* if we can enumerate points of each set by numbers 1, 2, $\ldots n$ and after that triangulate these sets in such a way that for every (elementary) triangle from the triangulation of S_1 one can find an (elementary) triangle in the triangulation of S_2 with the same numbers of vertices. In particular, the convex hulls of the sets S_1 and S_2 must have an equal number of vertices and be enumerated in the same way.

2.1. Prove that every triangulation of a n-point set has at most $3n-6$ edges (including the segments of the border of the convex hull).

2.2. Let the sets S_1 and S_2 have equal number of points and their convex hulls have an equal number of vertices. Suppose that the convex hull of S_1 has a vertex P such that all the points of the set $S_1 \setminus P$ are in a convex position. Prove that the sets S_1 and S_2 have a compatible triangulation.

2.3. Let the sets S_1 and S_2 have equal number of points and their convex hulls have an equal number of vertices. Suppose that the convex hull of each set contains at most

а) one;

б) two;

в) three

points. Prove that S_1 and S_2 have a compatible triangulation.

2.4. *An open problem.* Let the sets S_1 and S_2 be in a generic position, have equal number of points, and their convex hulls have an equal number of vertices. Is it true that S_1 and S_2 have a compatible triangulation?

2.5. Let T_1 and T_2 be two triangulations of a set S.

а) Prove that there exists a one to one correspondence (= a perfect matching) between the edges of the triangulations T_1 and T_2 such that the matched edges either cross or are identical.

б) Prove that there exists a perfect matching between the sets of triangles of T_1 and T_2 such that the matched triangles either overlap or are identical.

в) Prove that there exist a perfect matching between the sets of triangles of T_1 and T_2 such that the matched triangles have a common vertex and intersect at an inner point.

2.6. Consider a triangulation T of a set S that has the following property: every edge in this triangulation is locally minimal, i.e., it is shorter than every segment with endpoints in S that crosses it. Prove that T is the globally shortest triangulation (i.e., the sum of lengths of its edges less than the sum of lengths of edges of any other triangulation).

2.7. Let S_1 and S_2 be two n-point sets whose convex hulls are $(n-k)$-gons ($k \geqslant 2$) with vertices enumerated consequently by numbers 1, 2, $\ldots$, $n-k$. Prove that there exists a compatible triangulation of these sets that has at most $k-2$ additional vertices.

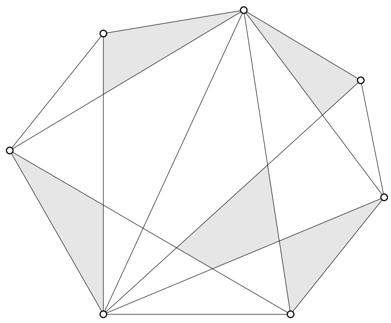

Рис. 5: For the problem 2.5.б: the shaded area is the intersection between two matched triangles.

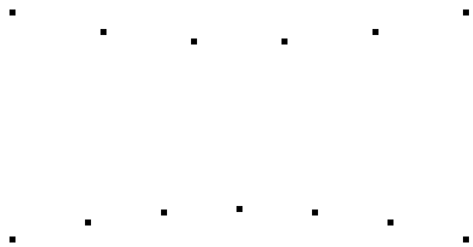

Рис. 6: For the problem 3.5.

3 Flipping edges of triangulations

3.1. A given polygon F can be cut into triangles by means of nonintersecting diagonals in two different ways. Prove that there are 4 vertices of F that form a convex quadrangle which is contained entirely in F.

Consider a triangulation of an arbitrary point set. *A flip* is the following transformation of this triangulation: choose a convex quadrangle consisting of 2 triangles of the triangulation, remove the diagonal that cuts it into two triangles, and draw the other diagonal.

3.2. Prove that every triangulation of a convex n-gon can be constructed from any other triangulation by at most

а) $2n-7$ flips;

б) $2n-10$ flips (if $n>12$).

в) (An open problem.) Find for sufficiently large n a convex n-gon and two its triangulations such that we need at least $2n-10$ flips to construct one triangulation from another. Is it true that such a polygon exists for any $n>12$?

3.3. Prove that we can obtain by flips any triangulation of a (not necessarily convex) polygon starting with any other of its triangulations.

The minimal number of flips that we need to obtain one triangulation from another is called *the distance* between the triangulations. For every n, let $W(n)$ denote the minimal number of flips that is sufficient to obtain any triangulation of an arbitrary polygon starting with any other of its triangulations.

3.4. Prove that there exists $C>0$ such that for all n we have $W(n)\leqslant Cn^2$.

Now we will describe a polygon that has two triangulations at a long distance.

3.5. Let $(a+b)$-gon $Q_{a,b}$ have a vertices $X_1, \ldots, X_a$, laying on a convex curve and b vertices $Y_1, \ldots, Y_b$, laying on a concave curve such that all the points X_i are above any line Y_jY_k, and all the points Y_i are below any line X_jX_k (see. fig. 6). Find two triangulations of this polygon which are at the longest distance from each other.

3.6. Prove that there exists $C>0$ such that for all n we have $W(n)\geqslant cn^2$.

4 *Number of triangulations*

We will consider triangulations as planar graphs.

4.1. Let T be a triangulation of a point set in the plane in a generic position. For every $i \geqslant 3$ let v_i denote the number of vertices of a degree i inside the convex hull of all points, and b_j, $j \geqslant 2$ denote the number of vertices of a degree j on the border of the convex hull. Prove that

$$n + 6 \leqslant 4v_3 + 3v_4 + 2v_5 + v_6 + 3b_2 + 2b_3 + b_4 .$$

Consider a triangulation T of some n-point set A, and let $p \in A$ be one of its points. We say that a triangulation T' is obtained by deleting a point p from T if all the segments of T that are not incident to p belong to T'. In the same case, we say that a triangulation T is obtained by inserting a point p into T'. Generally speaking, insertion and deletion are not uniquely defined.

4.2. а) Let a point $p \in A$ be inside of the convex hull of a set A, T' be an arbitrary triangulation of a set $A \setminus \{p\}$. Prove that there exist at most 9 triangulations which can be obtained by inserting a point p into triangulation T', and in which p has a degree 5.

б) Generalize this statement.

Let t_n denote the minimal possible number of triangulations of a n-point set, T_n be the maximal number of triangulations. This is a table of known values of t_n and T_n, and also Catalan numbers for $n \leqslant 10$.

n	3	4	5	6	7	8	9	10
t_n	1	1	2	4	11	30	89	250
c_{n-2}	1	2	5	14	42	132	429	1430
T_n	1	2	5	14	42	150	780	4550

Таблица 1: Number of triangulations for small n

In order to estimate Catalan numbers and other expressions containing factorials the Stierling formula is useful: $n! \sim \sqrt{2\pi n} n^n e^{-n}$ for $n \to \infty$.

4.3. Prove that $t_n \geqslant t_{n_1} t_{n_2}$, if $n_1 + n_2 = n + 2$, $n \geqslant 4$, $n_1, n_2 \geqslant 3$.

4.4. Prove that there exist numbers A, B, D, E, such that for $n \geqslant 5$

$$c_1 A^n \underset{(1)}{\leqslant} t_n \underset{(2)}{\leqslant} c_2 B^n , \qquad c_3 D^n \underset{(3)}{\leqslant} T_n \underset{(4)}{\leqslant} c_2 E^n .$$

By now these inequalities have been established for $A = 2.0129$, $B = 3.4641$, $D = 8$, $E = 59$.