Triangulations

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Problems

In these problems, we assume that all the sets of points on the plane (for example, vertices of polygons) are in a generic position.

1 Triangulations of polygons

1.1. Prove that an arbitrary polygon can be cut into triangles by means of diagonals that do not intersect in interior points.

There is a very popular number sequence in combinatorics — the Catalan numbers c_n ($n \geqslant 0$). It is defined by a recurrence relation

$$c_{n+1} = c_0 c_n + c_1 c_{n-1} + \ldots + c_n c_0 \quad (n \geqslant 0)$$

and "initial value" $c_0 = 1$. An exact formula for Catalan numbers is also known: $c_n = \frac{1}{n+1} C_{2n}^n$.

It is not too difficult to check that a convex n-polygon can be cut into triangles by c_{n-2} ways.

One of the other combinatorial realizations of Catalan numbers is a number of planted strictly binary trees (i.e., each vertex has either 2 or 0 sons (in the latter case this vertex is by definition a leaf) with $n+1$ leaves) (see fig. 1).

1.2. Find a bijective proof of the fact that the number of planted strictly binary trees with $n+1$ leaves is the Catalan number c_n.

1.3. In how many ways one can cut an arbitrary convex n-polygon into triangles such that each triangle contains an outer side of the n-polygon?

A triangulation of a polygon is defined as its partition into triangles by means of nonintersecting diagonals. Sometimes we also mean by a triangulation just the set of these diagonals.

Let $n > 3$ be a natural number. We say that triangulations T_1 and T_2 of n-polygons P_1 and P_2 are *compatible* if one can enumerate the vertices of these n-polygons by numbers 1, 2, …, n along the border in such a way that a diagonal connecting two vertices of P_1 belongs to the triangulation T_1 if and only if a diagonal connecting the vertices of P_2 with the same numbers belongs to the triangulation T_2.

It is evident that for arbitrary two n-polygons a compatible triangulation may not exist (see fig. 2)

Let several points be marked inside of an n-polygon. A partition of this n-polygon into triangles by means of nonintersecting segments with endpoints in the vertices of the n-polygon or in the marked points is called *a triangulation with additional vertices*. The additional vertices are also called Steiner points.

We say that two n-polygons P_1 and P_2 have *a compatible triangulation with k additional vertices* if we can mark k points in each n-polygon, enumerate the vertices of the n-polygons by numbers 1, 2, …, n along the border, enumerate the additional vertices by numbers $n+1$, $n+2$, …, $n+k$, and after that triangulate these n-polygons in a compatible manner, which means that for every segment in the triangulation of P_1 there exists a segment connecting vertices with the same numbers in the triangulation of P_2.

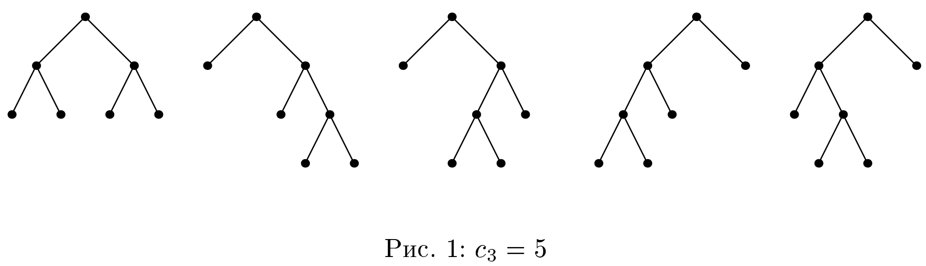

Рис. 1: $c_3 = 5$

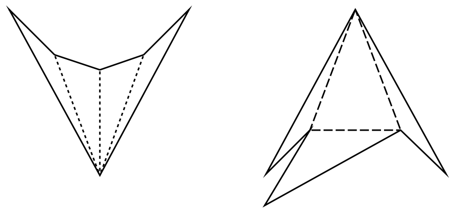

Рис. 2: Two polygons that can not be triangulated compatibly

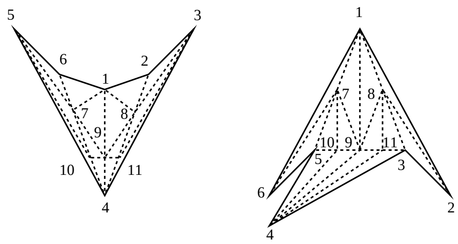

Рис. 3: A compatible triangulation

1.4. Every triangulation of a convex n-polygon necessarily contains a diagonal which cuts off a triangle. We will call such a triangle (or the vertex that was cut off) *an ear* of the triangulation.

а) Prove that every triangulation of a (not necessarily convex) polygon has at least 2 ears.

б) Prove that the number of triangulations of a convex n-polygon that have exactly k ears is equal to

$$\frac{n}{k} 2^{n-2k} C_{n-4}^{2k-4} c_{k-2} \,.$$

в) Prove that approximately a quarter of vertices of a polygon are in average ears of a triangulation. In other words, "an average" triangulation has about $n/4$ ears.

1.5. Consider a triangulation of a right n-polygon ($n \geqslant 5$) (possibly with additional vertices) such that all its triangles are

а) acute;

b) obtuse.

What is the least possible number of triangles which this triangulation can have?

1.6. Prove that all the triangulations of a convex n-polygon with k Steiner points have equal number of triangles and equal number of edges.

1.7. Prove that for any two n-polygons there exists a compatible triangulation with additional vertices (the number of additional points may depend on the polygons).

It occurs that there exists a universal triangulation with additional vertices.

1.8. а) Prove that for any $n > 3$ there exists k such that every n-polygon can be triangulated with k additional vertices and all the triangulations obtained in this way will be pairwise compatible.

б) Prove that there exists a universal triangulation in a stronger sense: for any n there exists k such that every triangulation of every n-polygon can be split by adding k additional vertices in such a manner that all these triangulations will be pairwise compatible.

1.9. Let $\kappa = \kappa(n)$ be the minimal k such that it is possible to triangulate with k additional vertices all the n-polygons so that all triangulations obtained will be pairwise compatible. Prove that there exist constants c_1, c_2, $c_3 > 0$, such that

а) $\kappa \leqslant c_1 n^2$;

б) $c_2 n^2 \leqslant \kappa$, if we assume that the order of vertices of every polygon is fixed;

в) $c_3 n^2 \leqslant \kappa$, if we assume that the order of vertices of every polygon is not fixed.

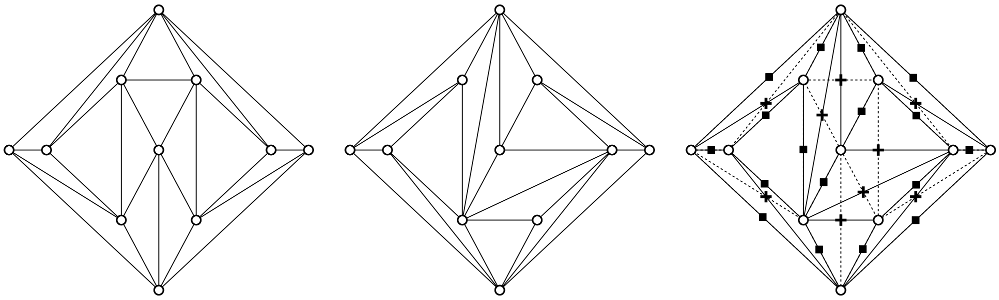

Рис. 4: For the problem 2.5.a: identical matched edges are marked by boxes, intersection points of two crossing matched edges are marked by crosses.

2 Triangulations of arbitrary sets

Let us remind that we consider the point sets in the plane in a generic position.

A triangulation of an arbitrary point set is by definition a partition of a convex hull of these points into triangles by segments that have endpoints in these points only.

We say that two sets S_1 and S_2 of n points each have *a compatible triangulation* if we can enumerate points of each set by numbers 1, 2, ... n and after that triangulate these sets in such a way that for every (elementary) triangle from the triangulation of S_1 one can find an (elementary) triangle in the triangulation of S_2 with the same numbers of vertices. In particular, the convex hulls of the sets S_1 and S_2 must have an equal number of vertices and be enumerated in the same way.

2.1. Prove that every triangulation of a n-point set has at most $3n-6$ edges (including the segments of the border of the convex hull).

2.2. Let the sets S_1 and S_2 have equal number of points and their convex hulls have an equal number of vertices. Suppose that the convex hull of S_1 has a vertex P such that all the points of the set $S_1 \setminus P$ are in a convex position. Prove that the sets S_1 and S_2 have a compatible triangulation.

2.3. Let the sets S_1 and S_2 have equal number of points and their convex hulls have an equal number of vertices. Suppose that the convex hull of each set contains at most

а) one;

б) two;

в) three

points. Prove that S_1 and S_2 have a compatible triangulation.

2.4. *An open problem.* Let the sets S_1 and S_2 be in a generic position, have equal number of points, and their convex hulls have an equal number of vertices. Is it true that S_1 and S_2 have a compatible triangulation?

2.5. Let T_1 and T_2 be two triangulations of a set S.

а) Prove that there exists a one to one correspondence (= a perfect matching) between the edges of the triangulations T_1 and T_2 such that the matched edges either cross or are identical.

б) Prove that there exists a perfect matching between the sets of triangles of T_1 and T_2 such that the matched triangles either overlap or are identical.

в) Prove that there exist a perfect matching between the sets of triangles of T_1 and T_2 such that the matched triangles have a common vertex and intersect at an interior point.

2.6. Consider a triangulation T of a set S that has the following property: every edge in this triangulation is locally minimal, i.e., it is shorter than every segment with endpoints in S that crosses it. Prove that T is the globally shortest triangulation (i.e., the sum of lengths of its edges less than the sum of lengths of edges of any other triangulation).

2.7. Let S_1 and S_2 be two n-point sets whose convex hulls are $(n-k)$-polygons ($k \geqslant 2$) with vertices enumerated consequently by numbers 1, 2, ..., $n-k$. Prove that there exists a compatible triangulation of these sets that has at most $k-2$ additional vertices.

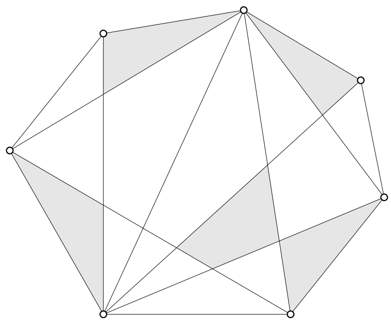

Рис. 5: For the problem 2.5.б: the shaded area is the intersection between two matched triangles.

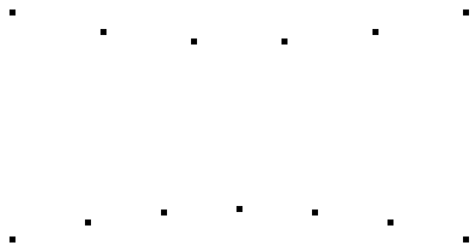

Рис. 6: For the problem 3.5.

3 *Flipping edges of triangulations*

3.1. A given polygon F can be cut into triangles by means of nonintersecting diagonals in two different ways. Prove that there are 4 vertices of F that form a convex quadrangle which is contained entirely in F.

Consider a triangulation of an arbitrary point set. *A flip* is the following transformation of this triangulation: choose a convex quadrangle consisting of 2 triangles of the triangulation, remove the diagonal that cuts it into two triangles, and draw the other diagonal.

3.2. Prove that every triangulation of a convex n-polygon can be constructed from any other triangulation by at most

а) $2n-7$ flips;

б) $2n-10$ flips (if $n>12$).

в) (An open problem.) Find for sufficiently large n a convex n-polygon and two its triangulations such that we need at least $2n-10$ flips to construct one triangulation from another. Is it true that such a polygon exists for any $n>12$?

3.3. Prove that we can obtain by flips any triangulation of a (not necessarily convex) polygon starting with any other of its triangulations.

The minimal number of flips that we need to obtain one triangulation from another is called *the distance* between the triangulations. For every n, let $W(n)$ denote the minimal number of flips that is sufficient to obtain any triangulation of an arbitrary polygon starting with any other of its triangulations.

3.4. Prove that there exists $C>0$ such that for all n we have $W(n)\leqslant Cn^2$.

Now we will describe a polygon that has two triangulations at a long distance.

3.5. Let $(a+b)$-polygon $Q_{a,b}$ have a vertices $X_1, \ldots, X_a$, laying on a convex curve and b vertices $Y_1, \ldots, Y_b$, laying on a concave curve such that all the points X_i are above any line Y_jY_k, and all the points Y_i are below any line X_jX_k (see. fig. 6). Find two triangulations of this polygon which are at the longest distance from each other.

3.6. Prove that there exists $C>0$ such that for all n we have $W(n)\geqslant cn^2$.

4 *Number of triangulations*

We will consider triangulations as planar graphs.

4.1. Let T be a triangulation of a point set in the plane in a generic position. For every $i \geqslant 3$ let v_i denote the number of vertices of a degree i inside the convex hull of all points, and b_j, $j \geqslant 2$ denote the number of vertices of a degree j on the border of the convex hull. Prove that

$$n + 6 \leqslant 4v_3 + 3v_4 + 2v_5 + v_6 + 3b_2 + 2b_3 + b_4 .$$

Consider a triangulation T of some n-point set A, and let $p \in A$ be one of its points. We say that a triangulation T' is obtained by deleting a point p from T if all the segments of T that are not incident to p belong to T'. In the same case, we say that a triangulation T is obtained by inserting a point p into T'. Generally speaking, insertion and deletion are not uniquely defined.

4.2. a) Let a point $p \in A$ be inside of the convex hull of a set A, T' be an arbitrary triangulation of a set $A \setminus \{p\}$. Prove that there exist at most 9 triangulations which can be obtained by inserting a point p into triangulation T', and in which p has a degree 5.

б) Generalize this statement.

Let t_n denote the minimal possible number of triangulations of a n-point set, T_n be the maximal number of triangulations. This is a table of known values of t_n and T_n, and also Catalan numbers for $n \leqslant 10$.

n	3	4	5	6	7	8	9	10
t_n	1	1	2	4	11	30	89	250
c_{n-2}	1	2	5	14	42	132	429	1430
T_n	1	2	5	14	42	150	780	4550

Таблица 1: Number of triangulations for small n

In order to estimate Catalan numbers and other expressions containing factorials the Stirling formula is useful: $n! \sim \sqrt{2\pi n}n^n e^{-n}$ for $n \to \infty$.

4.3. Prove that $t_n \geqslant t_{n_1} t_{n_2}$, if $n_1 + n_2 = n + 2$, $n \geqslant 4$, n_1, $n_2 \geqslant 3$.

4.4. Prove that there exist numbers A, B, D, E, such that for $n \geqslant 5$

$$c_1 A^n \underset{(1)}{\leqslant} t_n \underset{(2)}{\leqslant} c_2 B^n , \qquad c_3 D^n \underset{(3)}{\leqslant} T_n \underset{(4)}{\leqslant} c_2 E^n .$$

By now these inequalities have been established for $A = 2.0129$, $B = 3.4641$, $D = 8$, $E = 59$.

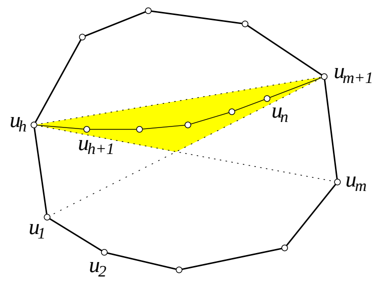

Рис. 7: Universal set S_1

5 *Four more problems*

2.8. Prove that the statements of problems 2.3. б) and в) is still valid if we suppose in addition that the numbering of vertices of the convex hulls of the sets S_1 and S_2 is fixed from the very beginning.

The next problem is a generalization of problem 2.2.

2.9. Let S_1 and S_2 be n-point sets whose convex hulls are h-polygons with vertices $u_1, \ldots, u_h$ and $v_1, \ldots, v_h$ in the counterclockwise order. Denote $u_{h+1}, \ldots, u_n$ the interior points of the set S_1. Suppose that there exists $m \leqslant h-2$ such that the set $S_1 \setminus \{u_1, u_2, \ldots, u_m\}$ is in convex position and moreover the interior points $u_{h+1}, \ldots, u_n$ are inside the triangle defined by the lines $u_h u_{m+1}$, $u_h u_m$ and $u_1 u_{m+1}$ (see fig. 7). Prove that the sets S_1 and S_2 have a compatible triangulation.

Finally the last problem generalizes problem 3.2.

3.7. A polygon $A_1 \ldots A_k B_{k+1} \ldots B_n$ is called *spiral* if its angles $\angle A_1, \ldots, \angle A_k$ are less than $180°$, and the angles $\angle B_{k+1}, \ldots, \angle B_n$ exceed $180°$.

Is it true that every triangulation of a spiral n-polygon can be constructed from any other one by $O(n)$ flips?

Solutions

1 *Triangulations of polygons*

1.1. It is sufficient to prove that every polygon has a diagonal that is contained inside the polygon.

If the polygon is convex then an arbitrary diagonal fits. If it is not convex consider its angle $\angle ABC > 180°$. Let us rotate a ray with the endpoint B from the position BA to the position BC, and every time let X denote the nearest from B intersection point of this ray and the border of the polygon. All the points X cannot belong to one side of the polygon because $\angle ABC > 180°$. Therefore, there exists a moment of time when the point X "jumps" from one side to another one. In this very moment the point X must be a vertex of the polygon and the diagonal BX is contained inside the polygon.

1.2. Consider a convex n-polygon $A_1A_2\dots A_n$ and one of its triangulations. The set of vertices of a tree consists of the centroids of all triangles ("interior vertices") and the midpoints of sides of the polygon except for A_1A_2 (leaves). The set of edges consists of segments of two kinds: 1) joining the centroids of adjacent triangles and 2) joining midpoints of sides with the centroid of the triangle that contains this side. Let the root be the triangle containing A_1A_2. We have obtained a planted flat binary tree. One can easily see that this correspondence is a bijection.

1.3. [13, problem 1.41] Answer: $n2^{n-5}$.

1.4. a) Given a triangulation, we construct a graph as follows. The set of vertices consists of the centroids of all triangles. The edges join the centroids of adjacent triangles. As in the previous solution, the graph thus formed is a tree. Ears correspond to the leaves of this tree.

б) [11] Fix a vertex v and consider a triangulation s.t. v is an ear of this triangulation. We denote the triangle containing v by the same letter. Let U be a tree with root v s.t. the set of its vertices is formed by triangles of the triangulation (two vertices are connected if the corresponding triangles are adjacent). Ears of the triangulations correspond to the leaves, and triangles with exactly one common side with the outline of the polygon — to the vertices of degree 2. Triangles with no common sides with the outline correspond to the vertices of degree 3. Note that the tree constructed carries an additional structure: if we travel along the edges starting at v and come to a vertex of the tree (i.e., a triangle) and go to another vertex increasing the distance from v, then we can come across either the left side of this triangle or across its right side. It follows that in the tree constructed each son is either left or right.

Note that a planted tree U carrying such "binary data" uniquely determines the triangulation.

Replace the tree U by a following planted binary tree. For each sequence of vertices, the first one and the last one of degree 1 or 3, and all other of degree 2, make the following "contraction": remove all the vertices of degree 2 (and the corresponding edges) and draw and edge from the vertex at the beginning to the vertex at the end of the sequence. Denote the graph obtained by U' (Fig. 8). Clearly, it is a planted binary tree (with root v). If the triangulation has k ears, the set of vertices of U' consists of k leaves and $k-2$ interior vertices (of degree 3), which sums up to $2k-2$ vertices in total.

Notice that

- there are c_{k-2} possible choices of (a planted binary tree) U' (it is one of the famous combinatorial definitions of the Catalan numbers);
- for each choice of U' the number of ways to "undelete" the remaining $n-2k$ vertices of degree 2 equals $\binom{n-4}{2k-4}$;

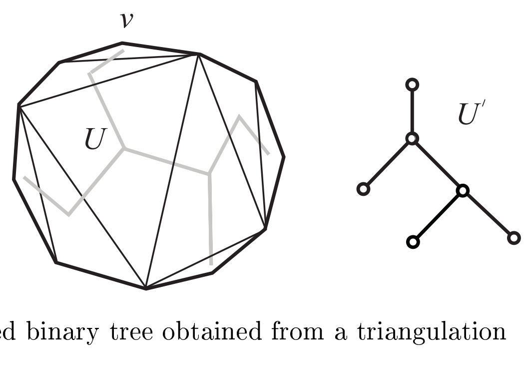

Рис. 8: Planted binary tree obtained from a triangulation

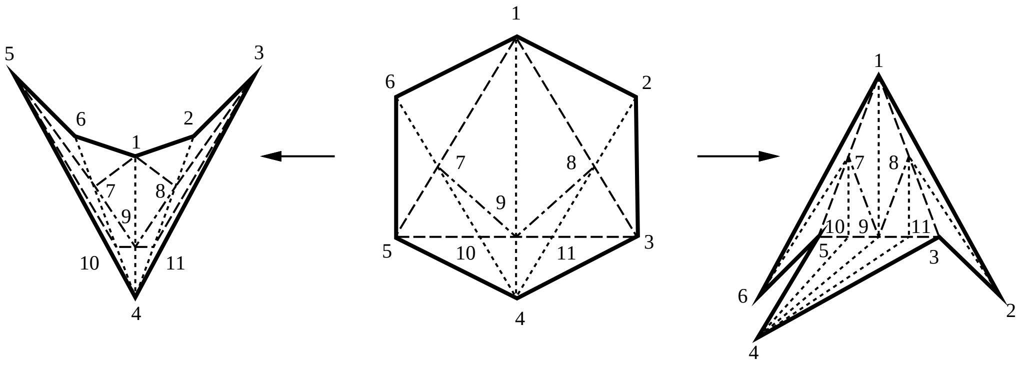

Рис. 9: Construction of compatible triangulations

Indeed, this problem is analogous to the following well-known problem: determine the number of ways to place m identical sweets in n different pockets. The answer here is $\binom{m+k-1}{k-1}$. Namely, consider a row of $m+k-1$ sweets and take $k-1$ of them. The holes split the row into k groups, and we place the sweets from the ith group to the ith pocket.

- the vertices being undeleted should be either left or right, so despite of the way of undeleting we have a choice between 2^{n-2k} variants;
- finally, there are n possible choices of the initial vertex v, and we will count each triangulation k times (each ear can be denoted by v).

It follows that the number of triangulations with exactly k ears equals $\frac{n}{k}2^{n-2k}C_{n-4}^{2k-4}c_{k-2}$.

в) [11] Consider the total number of ears in all triangulations. The contribution of each vertex to this number is c_{n-3}, since it is the number of triangulations for which this vertex is an ear. It follows that this number equals nc_{n-3}, and the average number of ears equals $\frac{nc_{n-3}}{c_{n-2}} = \frac{n(n-1)}{2(2n-5)}$.

1.5. This problem is from [2, problem 7.1]. The answer in both tasks is n triangles.

а) An example with n triangles is very simple: just connect the center of the n-polygon with all vertices. It is clear that for the number of triangles less than n there will exist a triangle with two common sides with the outline of the polygon, and so there exists an obtuse angle.

б) An example with n triangles can be constructed as follows. Choose a vertex of the polygon and plot all the diagonals containing this vertex. If n is odd, there is only one acute triangle in this triangulation. Split it into three obtuse triangles (adding one more vertex). In the case of even n, plot all the diagonals containing this vertex except for the diameter. Thus we have split the polygon into $n-4$ obtuse triangles and a quadrilateral. It remains to plot the shortest of the two diagonals of the quadrilateral and to split the remaining acute triangle into three obtuse triangles.

Suppose that there is an example with at most $n-1$ triangles. This means that there is a triangulation without additional vertices s.t. all triangles are obtuse. Consider the longest diagonal used in the triangulation. One of the triangles containing this diagonal cannot be obtuse.

For a triangulation with additional vertices the number of triangles is at least n (to prove that, calculate the sum of all angles in two different ways).

1.6. This problem is well known for those who are interested in olympiads. One of the popular questions is "Given k and n, how much triangles are there in the triangulation?". For example, consider problem 52 in [4].

To obtain the solution one can either sum up the angles in all triangles or use Euler's formula $V-E+F=2$.

1.7. One can find this statement and the figures 2, 3, 9 in [9].

Consider two n-polygons P_1 and P_2. Fix a clockwise numbering of vertices of each polygon. Consider another n-polygon P and a numbering of its vertices. Let T_i be a triangulation of P_i ($i=1,2$). Denote by T_i' a triangulation of P_i compatible with T_i. Using the triangulations T_i and T_i', define a piecewise linear mapping $\mathcal{L}_i\colon P \to P_i$, which identifies corresponding triangles of these triangulations. Namely, the vertices correspond to the vertices, and inside the triangle the mapping is determined from the linearity condition.

Plot all the diagonals used in T_1' and T_2' (see Fig. 9). The polygon P is split by these diagonals into several convex polygons. Split each of them into triangles and denote the resulting triangulations by S. The triangulations $L_i^{-1}(S)$ are compatible.

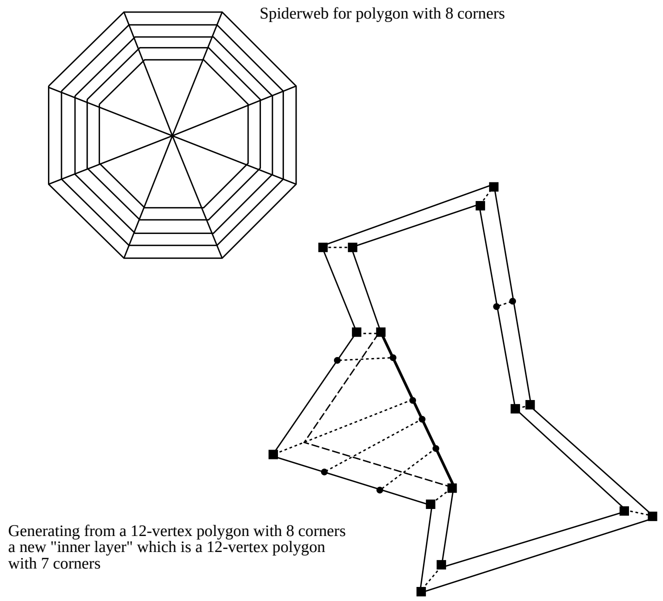

Рис. 10: A spiderweb and one iteration step

1.8. The previous construction allows to construct a universal triangulation. Namely, take a generic n-polygon, plot all its diagonals and split the polygons obtained into triangles. Given a n-polygon P_i, we can obtain its triangulation using the piecewise linear mapping L_i^{-1}. Thus the problem is solved.

1.9. a) To prove the upper bound $\kappa \leqslant c_1 n^2$ it is sufficient to introduce a universal triangulation with number of vertices behaving like a polynomial of degree 2 w.r.t. to n. Unfortunately, the number of vertices of the triangulation from the previous solution behaves like a polynomial of degree 4 w.r.t. to n.

Let us introduce a better construction of a universal triangulation (for a fixed numbering of vertices). In the case of a convex n-polygon this triangulation is a "spiderweb" comprising $n-2$ outlines of polygons similar to the initial one and n radii from the "center" (see Fig. 10). The quadrilaterals should be split into triangles (it is not done in the figure).

Given an arbitrary n-polygon, one can construct its triangulations using the following iteration process. Suppose that we already have a k-polygon P s.t. its outline contains n marked points (including the vertices). There exists a diagonal cutting off P a triangle Δ. Consider a polygon P' — a copy of P scaled a little and put into P. The set-theoretic difference $P \setminus P'$ is a "corridor" like a part of a spiderweb (see Fig. 10). Consider a triangle in P' corresponding to Δ and replace two "outer" sides of this triangle by a diagonal cutting it off P' (we need to place the necessary number of marked points on this diagonal). The polygon obtained has $k-1$ vertices. (Thus we described the step of the iteration.)

Proceeding like that, we will obtain a triangle s.t. its outline contains n marked points (including the vertices). Draw the radii connecting the "center" with these points. This construction requires $n(n-3)+1$ additional points.

б) Now we will introduce two n-polygons s.t. the necessary number of additional vertices for constructing compatible triangulations is cn^2. For the convenience we will suppose that n is a multiple of 4.

Consider Figure 11. Denote the leftmost triangle by P_1 and the rightmost triangle by P_2. Each of them consists of two parts with $n/4$ vertices connected with a corridor of "length" $n/4$. The numbering of the vertices of the second polygon is the first numbering shifted by $n/4$.

Consider two polygons P_1 and P_2 and their compatible triangulations T_1 and T_2. To prove that they involve cn^2 vertices, it is sufficient to prove that these triangulations consist of $c'n^2$ triangles.

As in problem 1.7, we use a piecewise linear mapping of one polygon onto another corresponding to the triangulations. Consider the vertical diagonals of P_1 (dashed on the Figure 11). The corresponding mapping transforms them into the polygonal lines lying inside P_2. Thus we have obtained $n/4$ non-intersecting polygonal lines in P_2, traversing P_2 from the upper side to the base. There are also $n/4$ horizontal segments (dashed on the Figure 11). Each polygonal line intersects each segment at least once (some materialization of this statement is plot on Figure 12), and so there are at least $n^2/16$ intersection points.

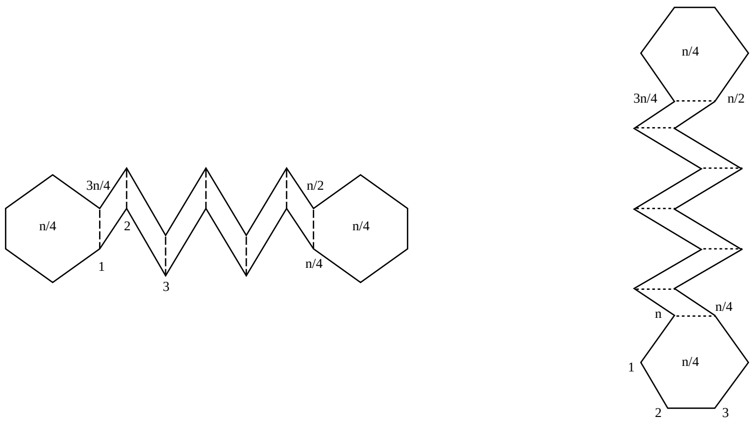

Рис. 11: cn^2 additional vertices

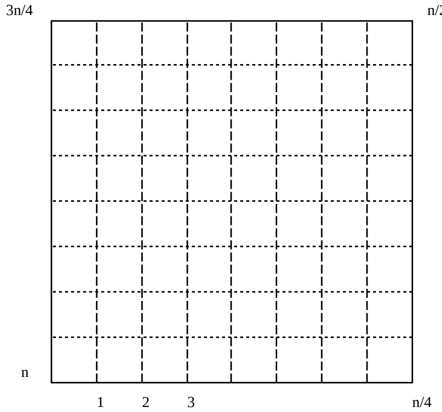

Рис. 12: Intersections of the polygonal lines with the horizontal segments

We will now show that every triangle in T_2 contains no more than 9 intersection points. First of all, notice that due to the construction of the snake-shaped corridor no edge of T_2 cuts more than two segments of the horizontal family. Similarly, no edge meets more than two vertical paths, for otherwise the corresponding edge of T_1 would meet three or more dashed chords, which is impossible. In particular, as any vertical path meeting a triangle has to meet at least two of its edges and every edge meets at most two vertical paths, only $(2 \times 3)/2 = 3$ vertical paths can enter our triangle. Thus at most 9 segment-path pairs can have points of intersection (remind that we use picewise-linear map between polygons, so an intersection of a segment/path and a triangle is necessarily a segment). Therefore, it would require at least $\frac{n^2}{16}/9 = \frac{n^2}{144}$ triangles to cover all intersections.

в) If the labeling of polygon corners is not given, then nevertheless it would require quadratic number of Steiner points in the worst case. Consider polygons on the fig. 13. Consider paths that correspond to vertical diagonals of the 1st polygon and short segments, that cuts three corridors onto quadrilaterals. Overlaying these two families produces quadratic number of intersections, no matter how polygon corners are matched.

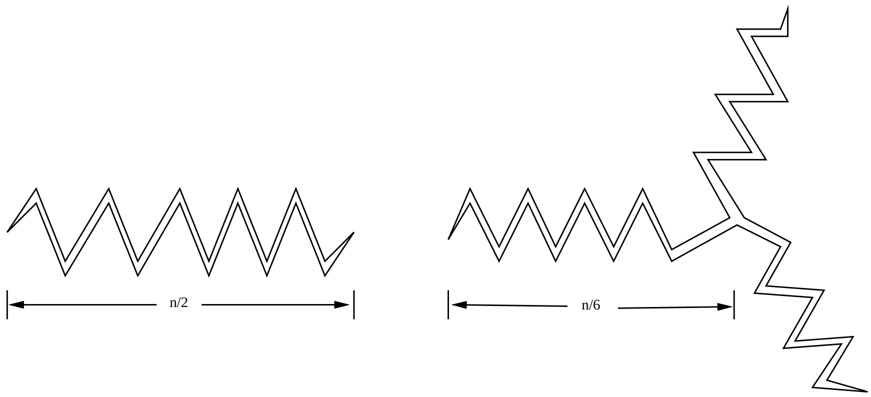

Рис. 13: When labelling is not given

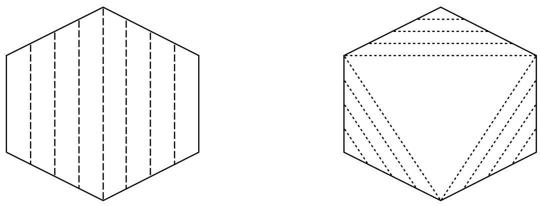

Рис. 14: Intersections of paths and segments

2 *Triangulation of arbitrary sets*

2.1. Euler's formula "$V - E + F = 2$" implies that "the more triangles a triangulation has the more edges it contains". Let us find an upper bound for the number of triangles. Sum of angles having the fixed vertex of the triangulation as a vertex is at most 2π. Hence the sum of all angles of all triangles does not exceed $2\pi n$. But in this calculation we have not taken into account that the sum of outer angles of the convex hull is at least 5π. It follows that we have at most $2n-5$ triangles. Hence we have at most $2n+n-6$ edges due to Euler's formula.

2.2. This problem together with fig. 15 is from [6].

Let us enumerate points of the set S_1. Let P have number 1, and other points be enumerated along the outline of the convex hull. Connect the point P with all such vertices of the convex hull of the remaining points that are visible from P. These points are exactly all interior points of the set S_1 and two more points from S_1 (see fig. 15). After that connect an arbitrary point of S_2 with all the interior points of S_2 and also with two its neighbor points in the outline of the the convex hull of S_2. Enumerate points of S_2 according to the numbering of S_1 and draw segments that correspond to the segments drawn in S_1. Now a part of the set S_2 that has not been triangulated yet is a polygon. Let us triangulate it arbitrarily. Then we can triangulate compatibly the corresponding part of the set S_1 because it is a convex polygon.

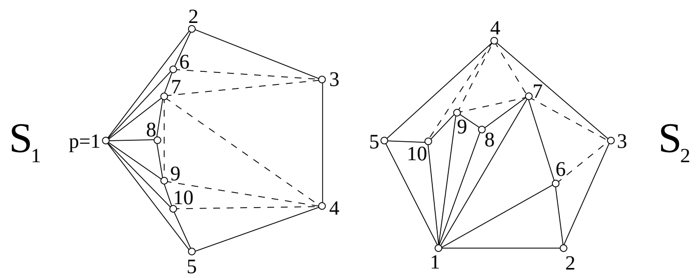

Рис. 15: A compatible triangulation

2.3. This problem is also from [6].

а) If both S_1 and S_2 have only one interior point then the star-like triangulation is compatible.

б) Let the sets S_1 and S_2 have two interior points. Connect one of the points in each set with all vertices. The second point belongs to one of the triangles thus obtained. Connect the second point with the vertices of this triangle. It is easy to see that we have constructed a compatible triangulations.

в) Let the convex hulls of the sets S_1 and S_2 contain n boundary points. Mark one interior point in each set S_1 and S_2 and denote it x. Make the following construction in both sets. Connect x with all vertices of the convex hull. We obtain n triangles in each set. It is clear that one of them does not contain other interior points. Consider the longest chain of consequent triangles in each set that does not contain other interior points.

Now choose the shortest of those chains. W.l.o.g. it is in S_1. Let it consists of ℓ triangles. Consider in S_2 a subchain consisting of ℓ triangles too. Let $u_1, u_2, \ldots, u_{\ell+1}$ denote the vertices of the convex hulls of each set which belong to the chosen chains; $u_{\ell+2}, \ldots, u_n$ denote other vertices.

Connect the point x in each set with the other two interior points (denote them y and z; see fig. 16) and draw the segments yu_1, yu_n and $zu_{\ell+1}$, $zu_{\ell+2}$.

Now the part of each set that is not triangulated yet is a polygon $yxzu_{\ell+2}\ldots u_n$. Moreover this polygon in S_1 is convex (because the only angles that can be greater than 180° are xyu_n or $xzu_{\ell+2}$, but in this case we could enlarge our chain of triangles by adding a triangle u_1xu_n or $u_{\ell+1}zu_{\ell+2}$ to this chain). Therefore, we can to construct a compatible triangulations.

2.4. The statement is true for the sets containing at most 8 points [6].

2.5. Something about Hall's marriage theorem. This problem together with fig. 4–5 is from [7].

а) Let the edges of the first triangulation be young men, the edges of the second triangulation be girls, we say that a young man and a girl are familiar if the corresponding edges coincide or intersect. We know from the problem 1.6 that the numbers of edges in both triangulations are equal. Let us check conditions of Hall's theorem.

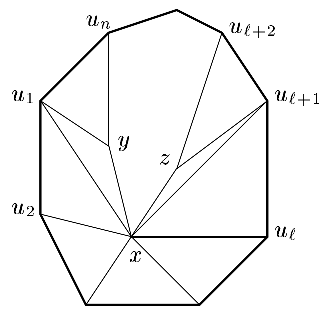

Рис. 16: For problem 2.3в.

Consider an arbitrary set Y of k young men, let G_1 be the set of girls which are familiar to them. We will check that G_1 contains at least k girls. Let G_2 be a complement of G_1 in the set of all girls. Observe that the edges from Y and G_2 do not intersect, hence the total number of edges in Y and G_2 is at most n. But the total number of edges in G_1 and G_2 is exactly n. Therefore G_1 contains not less elements than Y.

б) Analogously.

в) This time let young men and girls be triangles. Consider an arbitrary set Y of young men and a set G of girls that are familiar to them. Then for every vertex P of our triangulation the sum of those angles of triangles from Y that have P as vertex is not more than the sum of those angles of triangles from G that have P as vertex. If we will sum up these inequalities over all vertices P we will find that the sum of angles of all triangles from Y is not more than the sum of all angles of triangles from G. Hence G contains at least the same number of triangles as Y.

2.6. [7] The statement follows from problem 2.5.a. Indeed, there exists one-to-one correspondence between edges of our locally minimal triangulation and edges of any other triangulation, such that the corresponding edges either intersect or coincide. Then each edge from the locally minimal triangulation is not longer than the corresponding edge from the second triangulation. We obtain the statement of the problem by summing up these inequalities.

2.7. [5]. We prove by induction on k. Let $\sigma(k)$ denote the number of Steiner points that we need. For an arbitrary point set S and any of its boundary point S we define *an ear* $E(v)$ as a triangle whose vertices are v and two its neighbor vertices in the outline of the convex hull of S. We say that an ear is empty if it don't contain another points of S.

The base of induction is a statement of problem 2.3: $\sigma(2) = \sigma(3) = 0$.

Let $k \geqslant 4$.

Let $v_1, v_2, \ldots, v_{n-k}$ denote vertices of the convex hull of the set S_1, $w_1, w_2, \ldots, w_{n-k}$ denote the corresponding vertices of the convex hull of the set S_2; both sets of vertices are numbered in counterclockwise order. We may assume that for any i both ears $E(v_i)$ and $E(w_i)$ can not be empty (cut them and use induction otherwise). It is possible two cases.

1st case. Both ears $E(v_i)$ and $E(w_i)$ are nonempty for some i. Let $p \in E(v_i)$ be the first point that we meet when rotate a ray $v_{i-1}v_i$ around the point v_{i-1} counterclockwise. Choose a new Steiner point to the left of the ray v_{i-1} at a very small distance from the side v_iv_{i+1} such that a triangle v_ips and three adjacent triangles be empty (see fig. 17a). Analogous construction in the set S_2 provides points q and t. Now we can appply induction hypothesis to the sets $S_1 \cup \{s\} \setminus \{v_i, p\}$ and $S_2 \cup \{t\} \setminus \{w_i, q\}$. We decrease number of interior points on 1 by the price of 1 Steiner point.

2nd case. For every i exactly one ear $E(v_i)$ or $E(w_i)$ is nonempty. Consider a case $n-k \geqslant 5$, i.e. convex hulls of S_1 and S_2 are polygons with at least 5 vertices. (The case when these polygons are quadrilaterals we will consider later. Note also that these polygons can not be triangles in the situation under consideration.) Then one of the sets, say, S_1 have at least three empty ears. Choose two of them that do not intersect: E_{v_i} and E_{v_j}, $j \geqslant i+2$. We can find a triangulation T_1 of the convex polygon $v_1 \ldots v_{i-1}v_{i+1} \ldots v_{j-1}v_{j+1} \ldots v_{n-k}$ such that the ears E_{v_i} and E_{v_j} are on different sides with respect to each of its edges. Let T_2 be a compatible triangulation of the corresponding polygon in S_2. Consider a line ℓ_1 that moves through all the triangles in T_1 by the following manner:

— it starts from the position $v_{i-1}v_{i+1}$;

— when it moves through the triangle $v_av_bv_{b+1}$, it rotates around the point v_a (and hence intersects the edge v_bv_{b+1} every time);

— its final position is the edge $v_{j-1}v_{j+1}$.

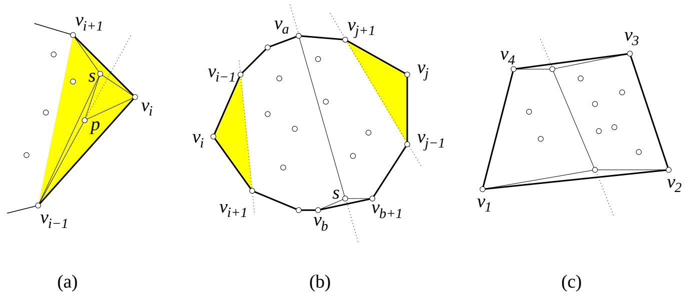

Рис. 17: Adding of Steiner points

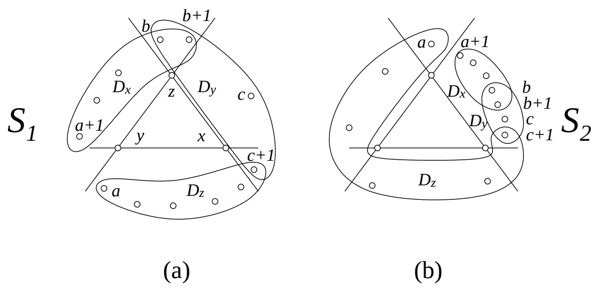

Рис. 18:

Consider the analogous line ℓ_2 for the triangulatuion T_2. Let α_i $(i = 1, 2)$ be the number of the interior points of the set S_i to the left of the line ℓ_i. When we start to move the lines, $\alpha_1 = 0$, $\alpha_2 \geqslant 1$. When we finish its movement, $\alpha_1 = k$, $\alpha_2 \leqslant k - 1$. Hence in some moment of time we have had an equality $\alpha_1 = \alpha_2$. Let the line ℓ_1 intersects the triangle $v_a v_b v_{b+1}$ in this moment. Then choose a new Steiner point s on the line ℓ_1 at a very small distance from the side $v_b v_{b+1}$ (see fig. 17b). Draw a segments connecting point S with v_a, v_b, v_{b+1} and analogous segments in S_2. We split the sets into two parts and can apply an induction hypothesis to these parts.

Now consider a case when the convex hulls of S_1 and S_2 are quadrilaterals. Choose a line ℓ_1 that intersects the segments v_1v_2 and v_3v_4 and such that there are exactly two interior points of the set S_1 to the left of this line (and $k - 2 \geqslant 2$ interior points to the right). We can find an analogous line in the set S_2. Choose two Steiner points on the line ℓ_1 at a very small distance from the sides v_1v_2 and v_3v_4 and construct two triangles as in fig. 17c. Fullfill an analogous constructions for the set S_2. Now apply the induction hypothesis to the obtained halfsets.

2.8. This problem together with 18, 19 is also from [6].

б) If both S_1 and S_2 have two inner points, draw a line through the inner points in both sets and enumerate the remaining points in each set. It is not hard to check that that there exist two numbers (say, a and b) such that the points enumerated by a and b in each set are in different halfplanes with respect to the corresponding line. Connect these point with the interior points. We have obtained two triangles of the future triangulations of S_1 and S_2. Proceed by connecting the inner points with the remaining points of the outline.

в) Let x, y and z denote the interior points of the set S_1. Let D_x be the subset of S_1 not on x's side of the line yz, we will call such a points *outer* with respect to yz. Define D_y and D_z analogously; see fig. 18a). The same notation we will use for the sets in S_2 which contain points with the same numbers. Now consider lines passing through the three interior points of S_2. We will say that some point is outer with respect to one of these lines if the point and the interior triangle are in the different halfplanes with respect to this line.

There are two possible cases of disposition of sets D_x, D_y and D_z in the convex hull of S_2.

1. The set D_x has a vertex outer with respect to one of three lines, the set D_y contain a vertex outer with respect to another line and, finally, the set D_z has a vertex outer with respect to remaining third line. Then consider an interior triangle and three adjacent triangles with vertices in the three points described above. The corresponding triangles in the set S_1 do not intersect by definition of sets D_x, D_y and D_z, Moreover, we can compatibly triangulate the remaining parts of S_1 and S_2 by connecting inner and boundary points as in previous solution.

2. Otherwise. In this case the subsets D_x, D_y and D_z of S_2 should dispose as in fig. 18(b). More precisely, two

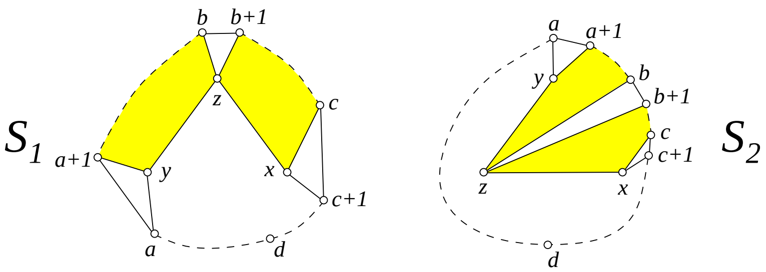

Рис. 19:

of them must be inside one of three angles defined by the three our lines. W.l.o.g. assume that these sets are D_x and D_y. Then consider a labelling of the interior points of S_2 as in fig. 19. We can assume that no two sets of D_x, D_y and D_z intersect. (For example, we can redefine these sets by removing intersection points of two sets from one of them. It is always possible except one only case when two sets contain a unique point P. But in this case the set $S_1 \setminus P$ is in convex position and the compatible triangulations exists due to problem 2.1.) Let D_x consists of points with numbers from $a+1$ to b, D_y consists of points with numbers from $b+1$ to c, and D_z consists of points with numbers from $c+1$ to a. Consider in both sets the edges shown on fig. 18. A polygon in S_1 with vertices z, y, $a+1$, ..., b is convex and hence has a compatible triangulation with the corresponding polygon in S_2. The same true for the polygon with vertices x, z, $b+1$,..., c. Therefore we need to construct a compatible triangulation of polygons $yzx(c+1)\dots d$. In S_1 possible reflex angles can occur at x or y. If this happens, say, for x, then there must exist a point d in $D_z \setminus D_y$ such that $\angle dxz < 180°$ and the segment xd does not intersect the edges that have been constructed yet. An analogous point exists for y. Therefore, we can compatibly draw at most two edges which split the poligon onto convex parts. For these parts and their dublicates in S_2 compatible triangulations can be easily found.

2.9. The proof of the results requires a preliminary lemma.

Lemma. Let P be a convex polygon with vertices v_1, dots, v_h in counterclockwise order, and let I be an arbitrary point set interior to P. Then for every choice of vertices v_i and v_j with $i \leqslant j-2$ there exists a simple path π of edges such that

(a) path connects v_i, I and v_j (in this order) and these are only vertices of π.

(b) the simple polygon, bounded by π and $v_i v_{i+1} \dots v_j$ can be triangulated without using an edge that connects two vertices of π.

We can apply this lemma to the convex hull of S_2 and its vertices v_h, v_{m+1}. The triangulation from the statement b) can be transferred to the polygon $u_h u_{h+1} \dots u_n u_{m+1} u_m \dots u_2 u_1$. After that for any triangulation of the remaining part of S_2 we can construct a compatible triangulation of the convex polygon

$$u_h u_{h+1} \dots u_n u_{m+1} u_{m+2} \dots u_{h-1}\,.$$

The proof of lemma is by induction on $d = j - i \geqslant 2$, the distance between v_i and v_j on P's boundary. For $d = 2$ the result is trivial: sort $I \cup \{v_i, v_j\}$ radially around v_{i+1} to produce a path star-shaped as seen from v_{i+1}, and triangulate by connecting all points in I to this vertex.

Let now $d > 2$. If the set I is totally to the right of $v_{i+2}v_j$ then connect v_i and v_{i+2} and slice off P the triangle $v_i v_{i+1} v_{i+2}$ Apply induction to the remaining convex polygon: the boundary distance between v_i and v_j is now $d-1$.

If, on the other hand, there are points of I to the left of $v_{i+2}v_j$ then let p be the one maximazing the angle $\angle v_i v_{i+1} p$ (see. fig. 20). Connect p with v_{i+1} and v_{i+2} and observe that the resulting triangle encloses no points of I. Now consider the convex polygon $Q = pv_{i+2} \dots v_j$ (shaded in fig. 20). Let I' and I'' be the subsets of I, exterior and interior to Q, respectively. By construction, no point of I' lies to the right of $v_{i+1}p$. So, as done for the case $d = 2$ above, construct the path for $I' \cup \{v_i, p\}$ that is star-shaped as seen from v_{i+1}, and build the coresponding triangulation. Finally, apply induction to the polygon Q whose boundary distance between p and v_j is $d-1$. This yields a simple path for $I'' \cup \{p, v_j\}$. The concatenation of two constructed pathes obviously is still simple and satisfies the statement of lemma.

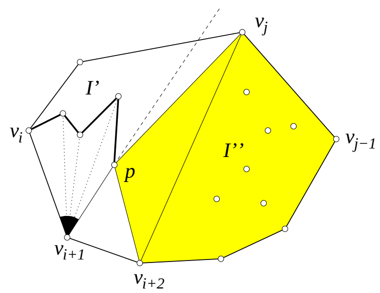

Рис. 20:

3 Flipping edges of triangulations

3.1. This problem was suggested for the Saint-Petersburg olympiad in 2002 by Yu. Lifshits. See [1], problem 77.

Use induction by the number of vertices. This statement is clear for a quadrilateral. Let us prove the induction step. Find a triangle ABC s.t. it has two common sides with the outline of F. (It is well known that such a triangle exists.) If the second triangulation also contains ABC, we can cut it off and use the induction hypothesis. In the other case, consider the triangles of the second triangulation containing the vertex B (namely, ABD_1, D_1BD_2, $\ldots$, D_kBC. How one can find a convex 4-polygon containing in F?

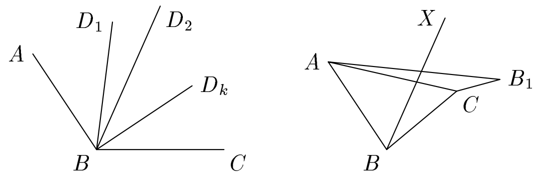

Рис. 21:

First idea (fig. 21, the left figure). Suppose that (at least) one of the angles $\angle AD_1D_2$, $\angle D_1D_2D_3$, $\ldots$, $\angle D_{k-1}D_kC$ is less than $180°$. Then the union of the two corresponding triangles is a convex quadrilateral. The assumption that all these angles are greater than $180°$, leads to a contradiction, since sum of angles $\angle D_1AB$, $\angle ABC$ and $\angle BCD_k$ is greater than $180°$ and hence angles of the polygon $CBAD_1D_2\ldots D_k$ sum up to a number greater than $(k+1)\cdot 180°$.

Second idea. Let us prove that one of the quadrilaterals $ABCD_i$ is contained in F. Consider the lines $\ell_1,\ldots,\ell_k$ s.t. ith line contains D_i and is parallel to AC. Let ℓ_m be the nearest one to AC. It follows that the diagonals BD_i $(1\leqslant i\leqslant k,\, i\neq m)$ intersect with ℓ_m at interior points, and so the quadrilateral BAD_mC is contained in the polygon $BAD_1\ldots D_kC$, hence in F.

Third idea (fig. 21, the right figure). We need existence of a vertex X inside the angle $\angle ABC$ s.t. the segment BX is contained in F (we can denote by X any of the vertices D_k). From now on we consider only the first triangulation. The segment AC is contained in a triangle ACB_1 with its vertex B_1 lying outside the angle $\angle ABC$ (if it is not true, it follows that B_1ABC is a convex quadrilateral). Without loss of generality we can assume that the segment AB_1 intersects with the ray BC. It is clear that AB_1 is not a side of F, because it intersects with the segment BX. It follows that this segment is a diagonal, hence it is contained in one more triangle (say, AB_1C_1).

It is clear that one of the segments B_1C_1 and AC_1 intersects with both sides of the angle $\angle ABC$ and with the ray BX (otherwise ACB_1C_1 is a convex quadrilateral). This segment also intersects with BX and hence is not a side of F. Proceeding like this, we will obtain that F has infinitely many vertices, and this is a contradiction.

3.2. This problem was suggested by D. Fomin for the autumn tour of the X Tournament of Towns ([3]). It is based on the results of [16].

a) Fix a vertex X. Suppose that for some triangulation T_1 there are k diagonals containing X. Then we can transform this triangulation into the triangulation T_X s.t. all its diagonals contain X using at most $n-3-k$ flips. To this end it is enough to make a flip that will increase the number of diagonals containing X. It is quite easy:

choose a diagonal XY which is not used in the triangulation. Let XA and XB be the diagonals (or sides) of the polygon that are the nearest ones to XY. Make a flip in the quadrilateral $AXBY$. It is clear that XY belongs to the resulting triangulation.

Now consider two triangulations T_1 and T_2. Choose a vertex X that is contained in at least one diagonal of T_1. It follows that we need at most $n-4$ flips to transform T_1 into T_X and at most $n-3$ flips to transform T_X into T_2. Hence we need at most $2n-7$ flips.

б) Acting as in the previous solution, find a vertex X s.t. the total number of diagonals with endpoint X in two triangulations is at least 4. Indeed, suppose that for each vertex this total number is at most three. It follows that the total number of diagonals in these triangulations is at most $3n/2$. Note that this number equals $2(n-3)$, which is greater than $3n/2$ for $n > 12$.

в) In [16] they prove that for all sufficiently large n there exist two triangulations s.t. the necessary number of flips equals $2n-10$. Still it is unknown whether it is true for all $n > 12$.

Here and below we call a diagonal which is contained in the interior of the polygon an *interior* diagonal.

3.3. Use induction by the number of vertices. For three vertices the statement is obvious. Let us prove the step of induction.

Consider the first triangulation T_1. Let ABC be its ear. Suppose that in the second triangulation T_2 there are exactly k diagonals containing B (say, BD_1, BD_2, …, BD_k; k may be equal to 0). Then the triangles ABD_1, $D_1BD_2, \ldots, D_{k-1}BD_k$, D_kBC are used in T_2 ($D_1, \ldots, D_k$ are all outside the triangle ABC since it is used in T_1). As in problem 3.1, it follows that one of the angles $\angle AD_1D_2$, $\angle D_1D_2D_3$, …, $\angle D_{k-1}D_kC$ is less than $180°$. If $\angle D_{i-1}D_iD_{i+1} < 180°$, then it is possible to make a flip in the quadrilateral $BD_{i-1}D_iD_{i+1}$. This flip removes the diagonal BD_i, so the number of diagonals of the second triangulation containing B decreases. Proceeding like this, we will replace the second triangulation with the one for which ABC is an ear. After that we can cut it off and use the induction hypothesis.

3.4. If you look carefully through the previous solution, you will see that the number of flips necessary to transform T_1 to T_2 does not exceed the number of *interior* diagonals of P.

Indeed, the number of flips necessary to make the second triangulation contain ABC does not exceed the number of interior diagonals with one endpoint at B. When we cut ABC off the polygon, we will forget about these diagonals and deal only with the remaining vertices.

Since the number of interior diagonals does not exceed the total number of diagonals, i. e., $n(n-3)/2 \leqslant n^2/2$, it follows that the statement of the problem is true for $C = 1/2$.

3.5. [12, theorem 3.8] Consider an arbitrary triangulation of the polygon $Q_{a,b}$. It contains triangles of two kinds: triangles adjacent to the upper broken line (convex) and ones adjacent to the lower broken line (concave). Let's mark every triangle of the first kind by the digit 1 and every triangle of the second kind by 0. We scan the sequence of triangles from left to right and obtain a sequence consisting of a unities and b zeros. Every flip interchanges two neighbor digits in this sequence. It is easy to see that any two sequences can be transformed to each other by means at most ab changes (move the first '1' to its place, then move next '1' to its place, and so on. Each of a steps requires at most b changes). From the other hand the sequence $11\ldots100\ldots0$ can not be transformed to $00\ldots011\ldots1$ using less than ab changes. Therefore, these two sequences (and triangulations corresponding to them) are at the longest distance from each other, and this distance equals ab.

3.6. For an even $n = 2k$ let's consider the polygon $Q_{k,k}$ from the previous problem. The distance between two its triangulations is $k^2 = n^2/4$, so $W(n) \geqslant n^2/4$. Hence the constant $c = 1/4$ satisfies the statement of this problem for an even number n.

As to the odd $n = 2k+1$, we can derive the inequality $W(n) \geqslant k(k+1) = (n^2-1)/4$ from the consideration of polygon $Q_{k,k+1}$. So in this case the constant $c = 1/5$ fits.

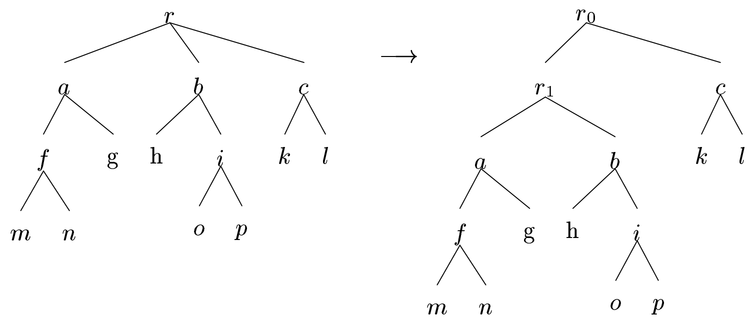

Рис. 22: The root of degree 3 is a result of contracting an edge.

4 Number of triangulatuions

4.1. [14] Let $v = \sum v_i$ be the number of the inner vertices, $b = \sum b_j$ be the number of the vertices in the outline of the convex hull of a given set, e and t be the number of edges and the number of triangles in the triangulation T. If we count triangles by edges, then we see that $3t = 2e - b$. By Euler's formula $t - e + b + v = 1$. Combining these two equalities we have

$$6v + 4b = 6 + 2e\,.$$

From the other hand, if we count edges via the number of vertices we see that

$$2e = \sum iv_i + \sum jb_j\,.$$

Therefore $6v + 4b = 6 + \sum iv_i + \sum jb_j$. Let us substitute $v = \sum v_i$ and $b = \sum b_j$ in this formula and add $n = \sum v_i + \sum b_j$ to both sides. Then remove the negative summands from the right hand side. We obtain the desired inequality.

4.2. а, б) [14] Let h_k denote the number of triangulations that we can obtain by inserting a point p of degree k. We will prove that

$$h_k \leqslant c_{k-1} - c_{k-2} = \frac{3}{2k-3} C_{2k-3}^{k-3}\,.$$

Let Δ denote the triangle of T' that contains the point p. After the inserting the point p in T' the union of k triangles that have p as a vertex will be a starshaped polygon Q with the “center” p. This polygon is a union of $k-2$ triangles of the triangulation T'. It does not contain other points of the set A in its interior. Conversely, every polygon with those properties provides a way of inserting p into triangulation.

Any triangle t of T' in such a polygon will be visible from p meaning that convex hull of $t \cup \{p\}$ contains no vertex of T' in its interior. Let G be the dual graph of T', whose nodes are the triangles in T' with two of them adjacent if the triangles share a common edge (we have used this graph in the solution of problem 1.4в). Let W be the subgraph in G induced by the triangles visible from p. This subgraph can not contain a cycle (because if we draw this subgraph placing its nodes in the centres of triangles then every cycle will contain a vertex of some triangle; this vertex will prevent one of triangles to be visible from p). Thus W is a forest. Let V be the tree in W that contains Δ; we may assume that Δ is a root of V. Then h_k coincides with the number of planted subtrees in V with $k-2$ nodes.

Note that V is a tree with root which has degree at most 3, and all of whose subtrees are binary trees. The number of $(k-2)$-node subtrees of V that contain the root is upperbounded by the number of such subtrees of the infinite tree whose root has exactly three children each of which is a root of infinite binary tree. We can interpret the root with degree three as a result of contraction of an edge in the infinite binary tree (see fig. 22). Then we are intrested in counting the number of those subtrees that contain the left child of the root (i.e. contain contracted edge) and have $k-1$ nodes. The number of those subtrees is clearly $c_{k-1} - c_{k-2}$.

For the point p that belong to the boundary of the convex hull of n-int set A we can analogously prove that $h_k \leqslant c_{k-2}$.

4.3. [8] Choose a line that pass through two of given points and have n_1 and n_2 points in the corresponding halfplanes (we assume that the points on the line belong to both halfplanes so $n_1 + n_2 = n + 2$). Any triangulation of the sets in the halfplanes can be extended to a triangulation of the whole set. But we cannot obtain every triangulation by this way of course.

4.4. 1. [8] It is not difficult to prove the inequality $t_n \geqslant \alpha^{n-2}$ for all n if it is true for small n. It follows from problem 4.3. Indeed, the sequence $u_n = \alpha^{n-2}$ satisfies all the recurrence relations of the form $u_p u_q = u_{p+q-2}$. If $t_n \geqslant u_n$, say, for $n = k, \ldots, 2k-3$, then the inequality holds also for $n \geqslant 2k-2$. This is true because for $m \geqslant 0$

$$t_{2k-2+m} \geqslant t_k t_{k+m} \geqslant u_k u_{k+m} = u_{2k-2+m}\,.$$

We can take $\alpha = \sqrt[3]{2}$. Then $t_n \geqslant \alpha^{n-2}$ for $n = 5, 6, 7$, as we can see from the table 4.

2. [14] In order to obtain an upper bound of t_n it is sufficient to find a set of n points that have small number of triangulations.

Consider a convex polygon $P = A_1 A_2 \ldots A_k$. Put a vertex B_i in the interior of P in a very small distance from the midpoint of the side $A_i A_{i+1}$. Let us count the number of triangulation of the obtained $2k$-point set.

Every triangulation of this set contains triangles $A_1 B_1 A_2, \ldots, A_k B_k A_1$. Hence, the number of triangulations of this set coincides with the number of triangulation of the polygon $Q = A_1 B_1 A_2 B_2 \ldots A_k B_k$.

Let B_i' be a point simmetrical to B_i with respect to a line $A_i A_{i+1}$ $(i = 1, \ldots, k)$, and Q' be a convex polygon $A_1 B_1' A_2 B_2' \ldots A_k B_k'$. It is clear that the number of triangulations of Q equals to the number of those triangulations of Q' in which no vertex B_i' is an ear. This number can be easily found by inclusion-exclusion formula. It equals $N = \sum\limits_{i=0}^{k} (-1)^i C_k^i c_{2k-i-2}$. It is possible to show that the inequality $N \leqslant 12^k$ holds. Thus we have an estimation $t_n \leqslant 12^{n/2} = \sqrt{12}^n$.

3. Consider a set of vertices of the polygon $Q_{n,n}$ (see problem 3.5). It consists of $2n$ points. Any triangulation of such set contains all the boundary of the convex hull and all segments between neighbor points on each of two convex curves. These "necessary" segments split the convex hull onto 3 parts — a polygon $Q_{n,n}$ and two more convex polygons. In order to triangulate this set we have to triangulate each of these three parts. Therefore the set has $(c_{n-2})^2 C_{2n-2}^{n-1}$ triangulations and we obtain an estimation $T_n \geqslant 2^{3n+O(\log n)}$.

4. We describe two construction of an upper bound.

[15] We will encode any triangulation by a binary string of $13n$ bits. This is provide an inequality with a constant $E = 2^{13}$. We use in the proof the statement of the four colors theorem.

We will encode a triangulation of n-point set S by a binary string of $f(n)$ bits, where the sequence $f(n)$ increases.

We will consider a triangulation as a planar graph. Due to four color theorem we can colour its vertices in four colors. This means that there exists an independent set I of vertices of size $n/4$. Remove this vertices along with the incident edges from the triangulation and retriangulate the "holes" thus created. This results in a triangulation T' of the set $S' = S \setminus I$. Now consider what information do we need to recover T if we know T'.

We assume that all our points are numbered and hence both points and edges are lexicographically ordered.

At first we can indicate which vertices were removed using n bits: let i-th bit keeps information whether the i-th was removed.

Secondly, we can indicate which edges were created by retriangulation using $9n/4$ bits. Indeed, due to the statement of the problem 2.1 any triangulation of a k-point set contains at most $3k$ edges. The triangulation T' contains $3n/4$ vetices. Therefore to encode any subset in the set of edges of T' we need at most $3n/4$ bits.

Hence to encode a triangulation of n-point set we need

$$f(n) \leqslant n + \frac{9n}{4} + f(\lfloor 3n/4 \rfloor)$$

bits. With the boundary condition $f(3) = 0$ we obtain inequality $f(n) \leqslant 13n$.

An article [15] contains also a hint how to improve slightly this upper bound. To do this we can use the fact that the set of $\lfloor 3n/4 \rfloor$ vertices we can choose by $C_n^{\lfloor 3n/4 \rfloor} \leqslant 2^{0.8113n}$ ways. Hence, it is enough $0.8113n$ bits instead of n bits.

Now describe the strongest upper bound [14].

Let $N(v, b)$ denote the maximal possible number of triangulations of the set which has b boundary and v inner points. We prove by induction that $N(v, b) \leqslant 59^v \cdot 7^b / C_{v+b+6}^6$.

The base of induction $b = 3$, $v = 0$: $59^0 \cdot 7^3 / C_{0+3+6}^6 = 49/12 \geqslant 1 = N(3, 0)$.

Now we prove the step of indultion. Let a set A have b boundary and v interior points. For each $i \geqslant 3$ let V_i denote a sum over all triangulations of A of the number of the interior vertices of degree i. For each $j \geqslant 2$ let B_j denote the sum over all triangulations of A the number of boundary vertices of degree j. Let N be the number of triangulations of A. Observe that deleting an interior point from A gives a point configuration with b boundary points and $v-1$ interiror points, while deleting a boundary point gives a point configuration with $v+b-1$ points at most v of which are interior. The number of triangulations of the latter set bounded from above by the number $N(v, b-1)$. The number of triangulations of the set A in which a certain vertex p has degree i is at most equal to the number of ways of inserting p with degree i in triangulations of $A \setminus \{p\}$. The inequality may be strict since insertions from different triangulations of $A \setminus \{p\}$ can lead to the same triangulation of A. Then the statement of problem 4.2a) implies that the number of ways h_5 to insert interior vertex of degree 5 does not exceed 9. Therefore,

$$V_5 \leqslant 9vN(v-1, b)\,.$$

Due to the estimates of the problems 4.2, we obtain that for the insertion of the interior vertex $h_3 = 1$, $h_4 \leqslant 3$, $h_6 \leqslant 28$; and for the boundary vertex $h_2 \leqslant 1$, $h_3 \leqslant 1$, $h_4 \leqslant 2$. Hence we conclude that

$$\begin{aligned} V_3 &\leqslant vN(v-1, b)\,, & B_2 &\leqslant bN(v, b-1)\,, \\ V_4 &\leqslant 3vN(v-1, b)\,, & B_3 &\leqslant bN(v, b-1)\,, \\ V_6 &\leqslant 28vN(v-1, b)\,, & B_4 &\leqslant 2bN(v, b-1)\,. \end{aligned}$$

Now we can use the inequality from the problem 4.1, which we sum up over all triangulations:

$$(6 + v + b)N \leqslant 4V_3 + 3V_4 + 2V_5 + V_6 + 3B_2 + 2B_3 + B_4 \leqslant 59vN(v-1, b) + 7bN(v, b-1)\,.$$

By induction hypothesis $N(v-1, b) \leqslant \frac{59^{v-1} \cdot 7^b}{C^6_{v+b+5}}$ и $N(v, b-1) \leqslant \frac{59^v \cdot 7^{b-1}}{C^6_{v+b+5}}$. Therefore,

$$(6 + v + b)N \leqslant (v + b)\,\frac{59^v \cdot 7^b}{C^6_{v+b+5}} = (6 + v + b)\,\frac{59^v \cdot 7^b}{C^6_{v+b+6}}\,.$$

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