Inscribed Links

Solutions of the problems suggested before the semifinal

1.1 Consider the graph consisting of 6 vertices and all joining them edges. Let each vertex correspond to a person. If two persons know each other, then paint red the edge joining the corresponding vertices, and paint it blue otherwise. Consider a vertex A. By the "pigeon holes principle" there exist three vertices X_1, X_2 and X_3 such that the edges AX_1, AX_2 and AX_3 have the same colour. Assume that they are red. If one of the edges X_iX_j $(i, j \in \{1;2;3\})$ is red, then all the sides of the triangle AX_iX_j are red, and the statement follows. Otherwise all the sides of the triangle $X_1X_2X_3$ are blue, and we are done.

1.2 *First Solution.* If the convex hull of the points has either four or five vertices, the claim is obvious: any two diagonals intersect. Consider the case when the convex hull is a triangle ABC. The two remaining points (say, D and E) belong to the interior of the triangle ABC. The point E belongs to the interior of one of the triangles DAB, DAC, DBC. We may assume that it belongs to DAC. Then EB crosses one of the segments DA and DC.

Second solution. The claim of the problem is a 2-dimensional analogue of Theorem 1 and can be proved analogously. We say that a set of points in the plane is *in general position* if no 3 points of this set are collinear. By *the total linking number* of 5-tuple of points in general position we mean the number of pairs of segments with their ends at these points intersecting at the interior point. It is easy to construct an example of a 5-tuple of points with the total linking number 1. After that it remains to show that the total linking number modulo 2 does not depend on the choice of 5 points. Consider two different 5-tuples. We can transform one of them into another, moving the points one by one (the motion can be assumed uniform). Clearly, while the points move leaving the 5-tuple in general position, the total linking number is constant. As one of the points (say, A) passes through a segment with ends at two of the remaining points (say, B and C), the resulting total linking number either differs by 2 from the initial linking number or is equal to it. Indeed, at such a moment the "state" changes (i. e., intersecting pairs of segments become disjoint and vice versa) for exactly two pairs: AD, BC and AE, BC (here D and E are the remaining points of the 5-tuple).

1.3 It is sufficient to show that if either $\operatorname{lk}(\Delta, \Delta') = 1$ or $\operatorname{lk}(\Delta', \Delta) = 1$, then the outline of the triangle Δ crosses the plane of the triangle Δ'. The first case is obvious. In the second case the outline of the triangle Δ' crosses the interior of the triangle Δ. It follows that the interior of the triangle Δ crosses the plane of the triangle Δ', hence the outline of the triangle Δ also crosses the plane of the triangle Δ'.

1.4 Let Δ and Δ' be two linked triangles. It is clear that condition 1) is satisfied. The condition 2) follows from task 1.3. Indeed, since the triangle Δ intersects the plane of the triangle Δ', it follows that the set $\Delta \cap l$ is not empty. By general position the set cannot consist of a single point. Let us prove that condition 3) is satisfied. Denote by $\{A, B\}$; $\{A', B'\}$ the sets $\Delta \cap l$ and $\Delta' \cap l$ respectively. All common points of the outline of the triangle Δ and the interior of the triangle Δ' belong to the segment $A'B'$. So exactly one of the points A and B belongs to the segment $A'B'$. This means that the pairs A, B and A', B' are linked in l.

Vice versa, let the pairs A, B and A', B' be linked in l. Then by the above the outline of the triangle Δ crosses the interior of the triangle Δ' at a single point, hence Δ and Δ' are linked.

1.5 According to the task 1.4, both of the conditions $\operatorname{lk}(\Delta, \Delta') = 1$ and $\operatorname{lk}(\Delta', \Delta) = 1$ are equivalent to the same condition.

1.6 It follows from task 1.4 that $\operatorname{lk}(\Delta, \Delta') = 1$ if and only if conditions 1)–3) are satisfied. None of them are changed under motion described. This needs checking only for property 1). This condition is not satisfied only if the planes of the triangles Δ and Δ' are parallel. Then by task 1.3 it follows that at this moment (and at all moments nearly before and after it) we have $\operatorname{lk}(\Delta, \Delta') = 0$.

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1.7 The following lemma may be useful:

Lemma. Let the vertex A of the triangle Δ move along a segment in space with constant speed, while two remaining vertices and Δ' stay put. Denote by Δ_t the position of the triangle Δ at the moment t, where $0 \le t \le 2$. Assume that the set of 6 vertices of Δ_t and Δ' is in general position for each t, except $t = 1$.

a) If $\Delta_1 \cap \Delta' = \emptyset$, then $\mathrm{lk}(\Delta_0, \Delta') = \mathrm{lk}(\Delta_2, \Delta')$.

b) If Δ_1 and Δ' have a unique common point distinct from any of their vertices, then the numbers $\mathrm{lk}(\Delta_0, \Delta')$ and $\mathrm{lk}(\Delta_2, \Delta')$ differ (by 1).

The proof of the lemma, the graph itself and tasks 1.8–1.10 are suggested as tasks after the semifinal.

2.1 The "most right" way to solve this task is to take a rope, make a knot from Fig. 4 on it, and try to deform it into one of the knots in Fig. 2, 3. The same is true for the links. It is convenient to represent the deformation of the rope by a sequence of pictures in which the consequent pictures are obtained one from another by a small deformation.

2.2 It suffices to show that a closed polygonal line with ≤ 5 sides is not knotted (i. e., represents a trivial knot). Clearly, a triangle is not knotted. Note that if in an n-polygon (i. e., a closed polygonal line with n sides) the interior of the triangle containing a pair of consequent sides does not intersect any of the remaining sides, then we can replace this pair of sides by the diagonal and obtain an $(n-1)$-polygon representing the same knot. Since in a 4-polygon there are no obstructions to such contraction, it follows that it contracts into a triangle and hence is not knotted. Assume that in a 5-polygon $ABCDE$ there is an obstruction to contraction of AB and BC. Then the side DE intersects the interior of the triangle ABC. But then AE does not intersect the interior of the triangle BCD, because they are separated by the plane ABC.

2.3 *Answers:* a) 6; b) 9; c) 6; d) 7; e) 8.

Example to a) is shown in Fig. 1. Another example: consider a regular triangle ABC in a horizontal plane, another triangle DEF with its vertex F placed outside the triangle ABC in its plane and the vertices D and E placed directly above and below the center of ABC respectively.

"Minimality". In a) and b) it is obvious. In c) it follows form task 2.2. Let us prove the "minimality" in d). It suffices to show that any two disjoint triangles in space either represent a simple link or are not linked (can be "unhooked"). If the outline of one of the triangles does not intersect the interior of the remaining triangle, then there is no obstructions to "unhooking". Otherwise each of the triangles intersects the interior of the other one at a single point. Consider the plane of one of the triangles (say, ABC). We may assume that it is horizontal. Let the side DE of the second triangle intersect the triangle ABC. Denote by F the remaining vertex of the second triangle. It is easy to deform the triangles ABC and DEF into the triangles of the above example to task 2.3.a.

The examples to b)–f), the "minimality" in f) and the whole task e) are the tasks after the semifinal.

2.4 a) Answer: 1, follows from Example 3.

b) Answer: 0, the solution is a task after the semifinal.

2.5 The proof of the statement is a task after the semifinal. Note that this statement may be useful in the solutions of the tasks from the first part of the problem.