Inscribed Links

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The first part of the problem leads to the proof of the following theorem.

Theorem 1. *(Conway–Gordon–Sachs, 1981) Suppose that there are given 6 points (in space) such that no 4 of them are coplanar. Then there exist two linked triangles with vertices at these points.*

The second part of the problem deals with generalizations of this theorem to arbitrary links (see the definitions and examples below). These two parts are almost independent, so you can solve the problem starting with either task 1.1 or task 2.1.

1. Linked triangles

1.1 Prove that in any company of 6 persons there are either three persons s.t. each of them is acquainted with others or three persons s.t. each of them is unknown to others.

1.2 Prove that for 5 coplanar points such that no 3 of them are collinear there exists a pair of segments with endpoints among these points intersecting at an interior point.

Definition of general position. We say that a set of points in space is *in general position*, if no 4 of them are coplanar.

Example 1. (Fig. 1) Consider a regular hexagon in a horizontal plane. The set of points A_1, A_2, A_3, A_4, A_5, A_6, placed directly above its vertices with heights 1, 2, 3, 4, 5, 6 respectively, is in general position.

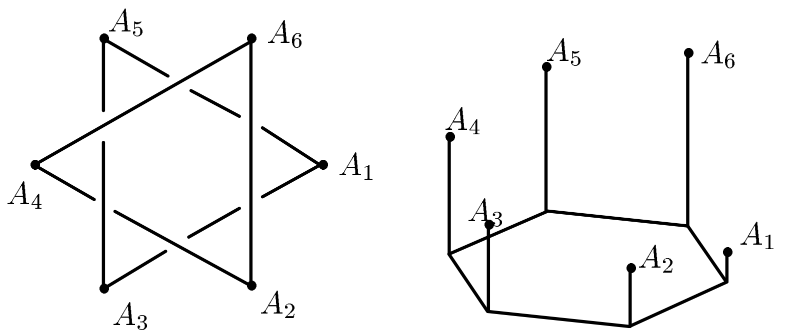

Fig. 1.

Example 2. The points with coordinates $(t; t^2; t^3)$ in a Cartesian coordinate system (for $t = 1, \dots, 6$, are in general position (try to check it yourself!).

Definition of the linking coefficient of two triangles. Let Δ, Δ' be triangles such that the set containing their 6 vertices is in general position. We say that these triangles are *linked*, if the outline of the triangle Δ intersect the interior of the triangle Δ' at a single point. Let us define the *linking coefficient* $\mathrm{lk}(\Delta, \Delta')$ of two triangles by the formula

$$\mathrm{lk}(\Delta, \Delta') = \begin{cases} 1, & \text{if } \Delta \text{ and } \Delta' \text{ are linked,} \\ 0, & \text{otherwise.} \end{cases}$$

Example 3. The triangles $A_1A_3A_5$ and $A_2A_4A_6$ from Example 1 are linked.

1.3 Suppose that the outline of one of the triangles Δ and Δ' does not intersect the plane containing the second triangle. Prove that $\mathrm{lk}(\Delta, \Delta') = 0$.

Definition of linked pairs of points. Suppose that there are given two pairs of points (say, two red points and two blue ones). in the same horizontal line. We say that these pairs are *linked*, if their order in the line is either (red, blue, red, blue) or (blue, red, blue, red) from the right to the left.

1.4 Prove that the triangles Δ and Δ' are linked if and only if the following 3 conditions are satisfied:

1) the planes of the triangles Δ and Δ' intersect at a line (denote it by l);

2) each of the sets $\Delta \cap l$ and $\Delta' \cap l$ consists of two points (here and below Δ and Δ' denote the outlines of the triangles);

3) the pairs of points $\Delta \cap l$ and $\Delta' \cap l$ are linked.

1.5 Prove that $\mathrm{lk}(\Delta, \Delta') = \mathrm{lk}(\Delta', \Delta)$.

1.6 Prove that $\mathrm{lk}(\Delta, \Delta') = \mathrm{const}$, if Δ moves leaving the set of 6 vertices of triangles Δ and Δ' in general position.

1.7 Suppose that the point A_1 of Example 1 moves upwards with constant speed. Denote by $A_1(t)$ its position at the moment t. Plot the graph of the function $\mathrm{lk}(A_1(t)A_3A_5, A_2A_4A_6)$.

Definition of the total linking number of a 6-tuple of points. Suppose that there are given 6 points in space in general position. We shall say that two triangles with (pairwise distinct) vertices at these points form a *divided pair*. *The total linking number* of a 6-tuple of points is the number of linked divided pairs.

Remark. For a 6-tuple of points, its total linking number is equal to

$$\sum \mathrm{lk}(\Delta, \Delta'),$$

where the sum is over all divided pairs (Δ, Δ').

1.8 Prove that the total linking number modulo 2 does not depend on the choice of a 6-tuple of points.

1.9 Construct an example of 6 points with the total linking number 1.

1.10 Prove Theorem 1.

2. Inscribed knots and links

Main notions. A *knot* is just a rope with its ends tied. We suppose it elastic, and do not distinguish between the knots obtained one from another deforming the rope.

The examples of *distinct* knots: trivial K, right and left trefoils T, figure eight E (Fig. 2).

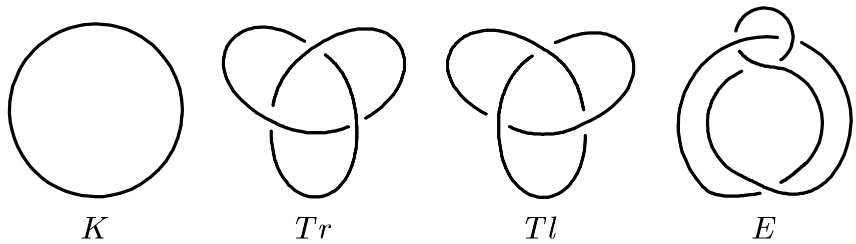

Fig. 2.

If there are several ropes, each having its ends tied, we call this figure a *link*.

The examples of *distinct* links: the Hopf link H (or simple link), double link D, the Whitehead link W, the Borromeo rings B (Fig. 3).

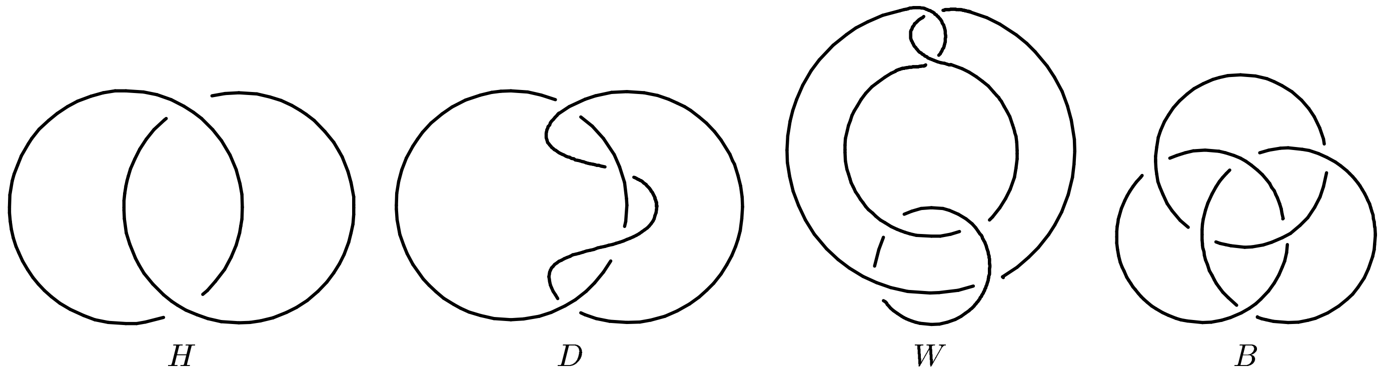

Fig. 3.

2.1 Into which of the knots and links from Fig. 2, 3 the knots and links from Fig. 4 can be deformed?

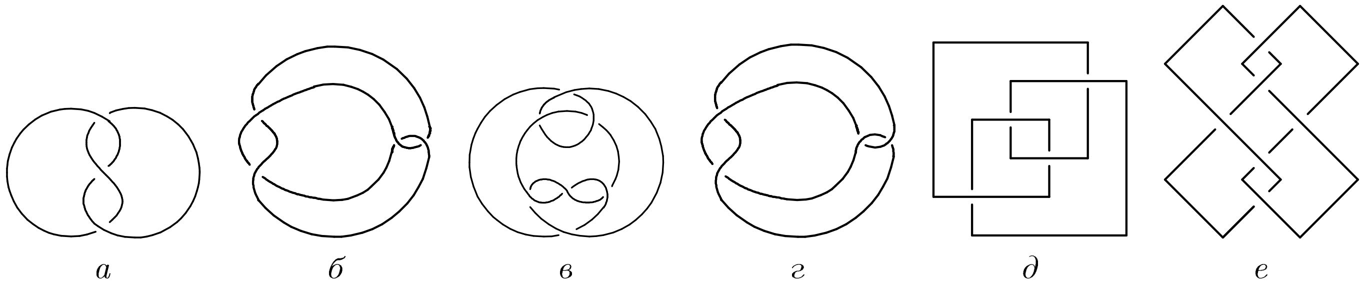

Fig. 4.

Definition of an inscribed knot and link. Let M be the set of points in space, A be a knot. Suppose that the rope of M is deformed into a polygonal line A' with vertices belonging to M. Then we say that (the knot) A' is *inscribed* into M. We also say that A' *represents* A. An *inscribed* link can be defined analogously.

Remark. Any knot inscribed into M is a subset of the figure obtained from M by joining all points by segments of straight lines.

Theorem 2. *(Negami, 1991) For any knot there exists a positive integer N such that this knot can be inscribed into any set of N points in space such that no 4 points of this set are coplanar.*

(Try to prove this theorem for the knot trefoil. Some hints to the proof of this theorem and some related open questions will be given in the tasks introduced after the semifinal.)

2.2 Prove that any knot inscribed into the set consisting of ≤ 5 points is trivial.

2.3 Find the minimal set into which the following knots and links can be inscribed

a) simple link;

b) the Borromeo rings;

c) trefoil;

d) double link;

e) figure eight;

f) the Whitehead link.

(We do not ask to prove the minimality).

2.4 Find the linking coefficients:

a) of two triangles representing a simple link;

b) of two triangles among three triangles representing the Borromeo rings.

2.5 Let Δ, Δ' be two triangles in space, $\overline{\Delta}, \overline{\Delta'}$ be their projections onto a horizontal plane. Assume that no 3 of 6 vertices of $\overline{\Delta}$ and $\overline{\Delta'}$ are collinear. For each point of $\overline{\Delta} \cap \overline{\Delta'}$ (*a cross-road*) draw which of the sides is higher (Fig. 1). Prove that $\mathrm{lk}(\Delta, \Delta')$ is equal modulo 2 to the number of cross-roads s.t. the side of $\overline{\Delta}$ is higher than the side of $\overline{\Delta'}$.