The problems suggested after the semifinal

(The solutions do not use the results of the previous tasks)

2. Inscribed knots and links

Definition of a knot (link) inscribed into a coloured set. Consider a set of red and blue points in space. By *an inscribed knot* we mean a polygonal line (representing some knot) such that the endpoints of each side are different-coloured. *An inscribed link* is defined analogously (e. g., see Theorem 1').

Theorem 2'. *(Miyauchi, 1994) For any knot (link) there exists a positive integer N such that this knot (link) can be inscribed into any set of N red and N blue points in space such that no 4 points of this set are coplanar.*

(Clearly, the Negami theorem follows from the Miyauchi theorem).

Definition of a positive set. Consider a set of N red and N blue points in space in a general position. Join different-coloured points pairwise by segments. Draw the projection of the obtained figure (onto the horizontal plane). Further we work only with plane projections, so we omit the word 'projection' from the phrases 'projection of a point of the set' and 'projection of a segment' if it does not leave to a confusion.

Assume that no interior points of the segments drawn belong to the set considered. For each intersection point of the segments (*a cross-road*) draw which of the segments is higher (as in task 2.5). Let us move along the projection of the higher segment from its blue endpoint to the red one. If the blue endpoint of the second segment is on the left side, then the cross-road is called *positive* (see the left part of Fig. 1'). Otherwise it is called *negative* (see the right part of Fig. 1').

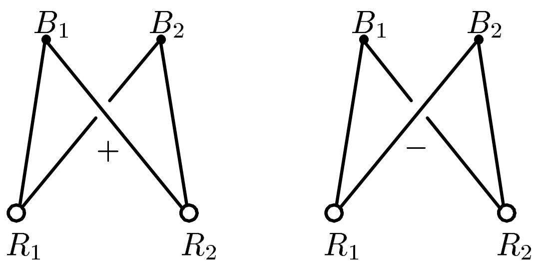

Fig. 1'.

The initial set of points is *positive*, if for any two blue points B_1, B_2 and any two red points R_1, R_2 either the segments B_1R_1, B_2R_2 or the segments B_1R_2, B_2R_1 intersect at an interior point, and the corresponding cross-road is positive. The *negative* set is defined analogously.

Example 1'. An example of a positive set is shown in Fig. 2.

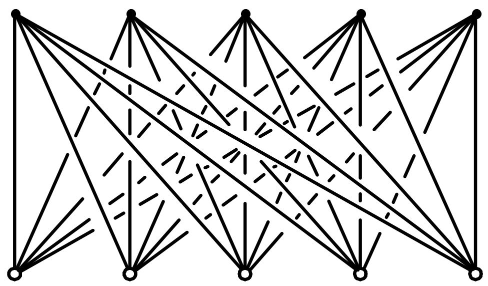

Fig. 2'.

2'.1 Paint red and blue the set of Example 1 to make it positive.

We assume that in tasks 2'.2–2'.5 there is given a positive set with $N \geq 2$.

2'.2 All blue points belong to the same halfplane bounded by a line joining a pair of red points.

Definition of the order '>'. Consider a positive set. Let R_1 and R_2 be two red points. Let us move along the line R_1R_2 from R_1 to R_2 (assume this line horizontal). We say that $R_1 > R_2$ if all blue points stay on the left during this motion.

2'.3 Prove that if $R_1 > R_2$ and $R_2 > R_3$, then $R_1 > R_3$.

2'.4 Prove that if the segment R_1B_1 is higher than R_2B_2, then $R_1 > R_2$.

2'.5 There exists a numbering of red and blue points such that for any $i < j$ and $k \neq l$ the segment R_iB_k is higher R_jB_l.

2'.6 Inscribe a trefoil into the set from Example 1'.

2'.7 A trefoil can be inscribed into any positive set of 5 blue and 5 red points.

2'.8 (the Ramsey theorems) a) Prove that for any m and n there exist $R(m,n)$ such that if we paint red and blue the edges of a complete graph with $R(m,n)$ vertices, then there exist either a red complete subgraph with m vertices or a blue complete subgraph with n vertices.

b) State and prove the generalization of a) to the case of 3 colours.

c) State and prove the generalizations of a) and b) to complete bipartite graphs.

d) A coloured set of points in space in a general position is called *neutral*, if no two projections of segments with different-coloured endpoints intersect (at an interior point). Prove that for any n there exist $R(n)$ such that any set of $R(n)$ blue and $R(n)$ red points in space in a general position contains either a positive or negative or neutral subset of n blue and n red points.

e) Prove that for any n there exist $R(n)$ such that any set of $R(n)$ blue and $R(n)$ red points in space in a general position contains either a positive or negative subset of n blue and n red points.

2'.9 Prove the Miyauchi theorem and the Negami theorem for a trefoil.

2'.10 Inscribe the knots and links from Fig. 2, 3 in a positive set and prove the Miyauchi theorem and the Negami theorem for them.