"12"

Problem proposed by V. Prasolov, M. Skopenkov and S. Dorichenko

Introductory problems

Let us consider a convex polygon $M = A_1A_2 \dots A_n$ all of whose vertices lie in the lattice of points with integer coordinates (Figure 1 on the left). Suppose that O is the only lattice point in the interior of M. Draw vectors $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \dots \overrightarrow{A_nA_1}$ from the point O and for any obtained segment choose its lattice point nearest to O. Connecting consecutive chosen n points we get a polygon M^*, so called *dual* to the original polygon (Figure 1 on the right).

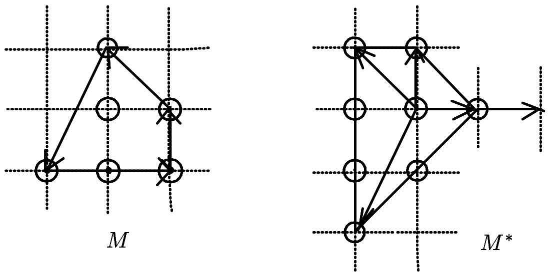

Figure 1.

This series of problems is devoted to a proof of the following theorem discovered only few years ago:

Theorem on 12. *Suppose that O is the only lattice point in the interior of a convex polygon M, the boundary of M contains b lattice points, and the boundary of M^* consists of b^* lattice points. Then*

$$b + b^* = 12.$$

0. Draw the polygons dual to polygons on Figure 2. How many lattice points are in the interior of M^* in each case? What is M^{**}? Is there a relation between areas of M and M^*? Is it possible that $M = M^*$? Formulate your observations and hypothesis and try to prove them.

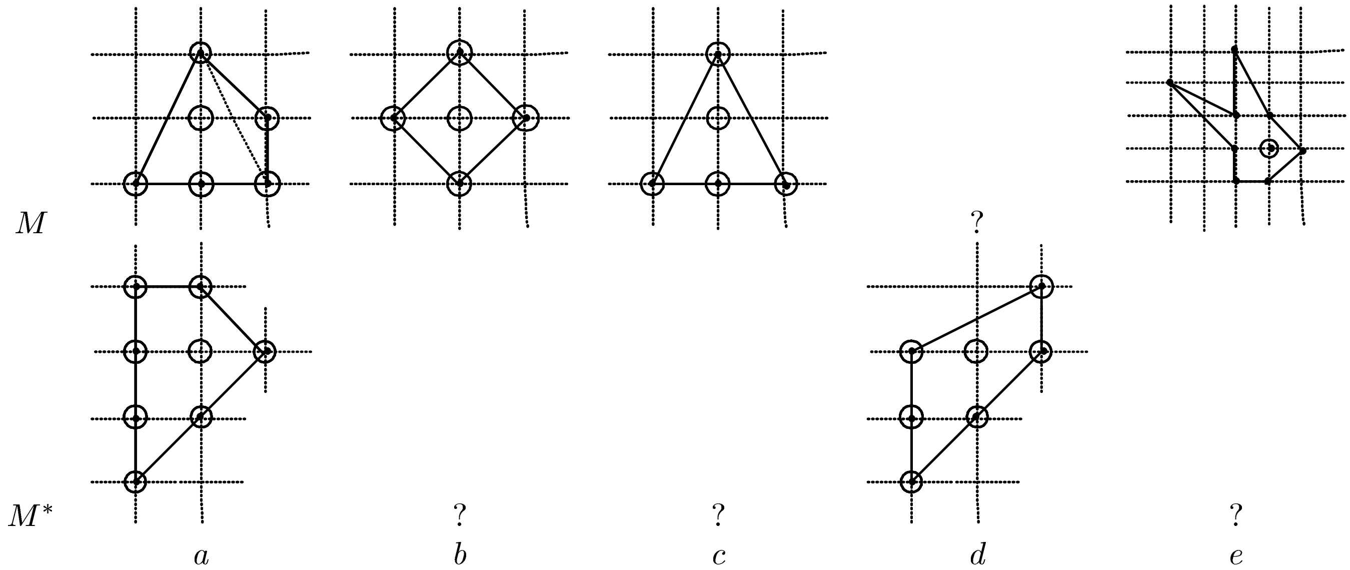

Figure 2.

1. The interior of a triangle contains exactly one lattice point. What is the maximal number of points on its sides?

2. Prove Theorem on 12 for a parallelogram with $b = 4$.

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A simple triangle is a triangle without lattice points in the interior and on the boundary (excluding vertices). For a triangle ABC *a jump* is the operation shown in Figure 3: we replace the vertex A by the point symmetric to A with respect to B.

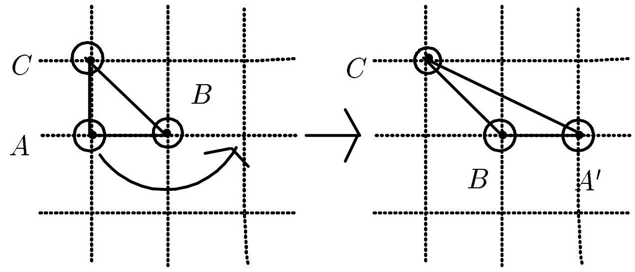

Figure 3.

3. a) The area of a triangle is invariant under jump.

b) For a simple triangle after a jump we obtain a simple triangle.

c) A simple triangle has no more then one obtuse or right angle and the lastcase is possible only for the triangle having sides 1, 1, $\sqrt{2}$; let us call such a case *minimal*).

d) From any non-minimal triangle one can obtain by a single jump a triangle whose greatest side is less then the greatest side of the original triangle.

e) From any simple triangle one can obtain a minimal triangle by a finite number of jumps.

f) The area of any simple triangle is equal to $1/2$.

4. a) If a convex quadrilateral does not contain lattice points in the interior and on the boundary (excluding vertices) then this quadrilateral is a parallelogram.

b) Is there a convex pentagon without lattice points in the interior?

c) In the interior of a convex polygon there is only one lattice point. What is the maximal number of its vertices?

5. a) The boundary of a triangle does not contain lattice points (excluding vertices) and there is only one interior lattice point. Then this point is the centroid.

b) Prove Theorem on 12 for a triangle with $b = 3$.

Let S be the area of a polygon. Suppose that its boundary contains b lattice points and its interior contains i lattice points.

6. a) Let us consider a triangulation of the polygon with vertices at $b + i$ given lattice points. How many triangles may have this triangulation?

b) Prove *Pick's formula:*

$$S = i + \frac{b}{2} - 1.$$

7. Let the ratio of the area of a polygon to the square of one of its sides is irrational (it is true, for example, for equilateral triangle). Prove that one can not draw this polygon (or a polygon similar to it) so that all its vertices are lattice points.

8. The king passed round the chess-board 8×8 squares passing once through any square and returned back with the last move. The broken line connecting centers of squares through which passed the king is not self-intersecting.

a) Find the maximal area bounded by this broken line.

b) Find the maximal length of the broken line.

Deleting a triangle is cutting off from polygon M a simple triangle having two common sides with M. For example, from the polygon on Figure 2a one can obtain the polygon on Figure 2b by deleting a triangle. The reverse operation is called *adding a triangle*.

9. Is it true that every 2 polygons

a) in Figure 2; b) with no lattice points inside; c) which are convex and contain 1 lattice point inside; can be obtained one from another by deleting and adding triangles?

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Basic problems

Let O be the origin of Cartesian coordinate system on the plane. Let us assign to a point $A(a,b) \neq O$ the line A^* given by the equation $ax + by = 1$. This line is called *dual* to the point. And vice versa to any line $ax + by = 1$ assign *dual* point (a, b).

10. Draw configurations dual to configurations on Figure 4. Verify that $l^* \in A^*$ on Figure 4a and lines A^*, B^*, C^* on Figure 4b have a common point. Prove that forany point $C \neq O$ the dual line C^* is ortohonal to OC and goes through the point $C' \in OC$ such that $OC' \cdot OC = 1$.

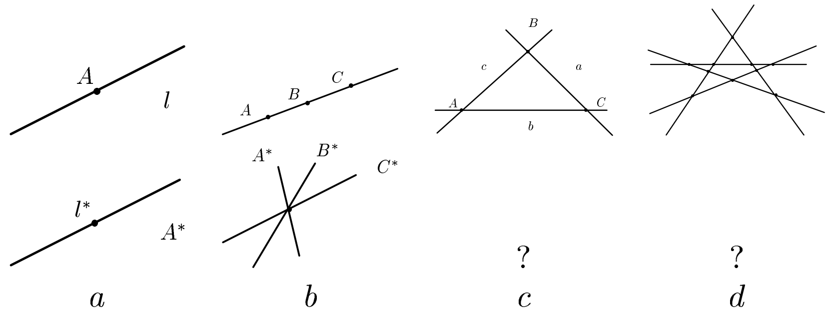

Figure 4.

For a polygon $M = A_1 \dots A_n$ we set $M^* = (A_1A_2)^* \dots (A_nA_1)^*$.

11. Verify that this definition of the dual polygon is compatible with the previous definition (up to a rotation by $90°$). Prove that $M^{**} = M$.

12. If the interior of M contains exactly one lattice point, then the interior of M^* also contains exactly one lattice point.

13. a) Deleting a triangle from M implies adding a triangle to M^*.

b) From any convex polygon with exactly one lattice point in the interior one can obtain a parallelogram with $b = 4$ using deleting/adding triangles.

c) Prove Theorem on 12.

14. Find the figure dual to the non-convex polygon on Figure 2e. Under which assumption on M both definitions of M^* are equivalent? How to generalize Theorem on 12 for non-convex polygons?

Let γ be a curve. The *dual* curve γ^* is the set of all points dual to all tangent lines to γ.

15. Draw the curve dual to the unit circle. Complete Figure 5. Is it true that $\gamma^{**} = \gamma$?

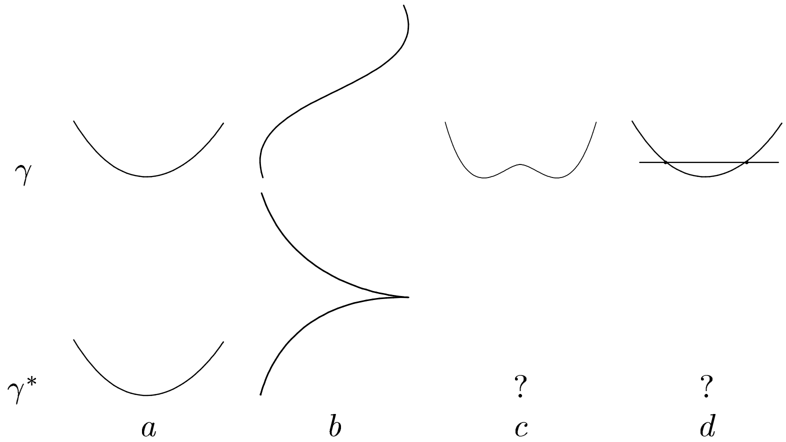

Figure 5.

16. (Open problem.) How to generalize Theorem on 12 for polygons with several lattice points in the interior?

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