Supplementary problems

17. One need to draw a broken non self-intersecting line whose vertices are all lattice points in the interior of a $p \times q$ rectangle and whose edges are either vertical or horizontal.

a) Find p and q such that it is possible.

b) Find the length of this line.

c) Find the area bounded by this line.

18. a) Denote by kM a polygon obtained from M by the homothety (dilation) with the coefficient k and the center in the origin. Prove that $2S(M) = n(2M) - 2n(M) + 1$, where $n(M)$ is the number of lattice points in the interior of M and on the boundary.

b) Find and prove an analogous formula for the volume of a polyhedron in space.

19. Find an upper border to the number of boundary lattice points of a convex polygon with n lattice points inside.

An affine transformation is the transformation of the plane given by the formula

$$\begin{cases} x' &= ax + by, \\ y' &= cx + dy, \end{cases}$$

preserving the lattice of integer points (this means that a, b, c, d are integers and $ad - bc = 1$).

20. Classify all the polygons

a) without lattice points in the interior; b) with exactly 1 lattice point in the interior;

up to an affine transformation.

21. a) Find the image of the line $y = ax + b$ under the transformation $(x, y) \mapsto (\frac{x}{y}, \frac{1}{y})$.

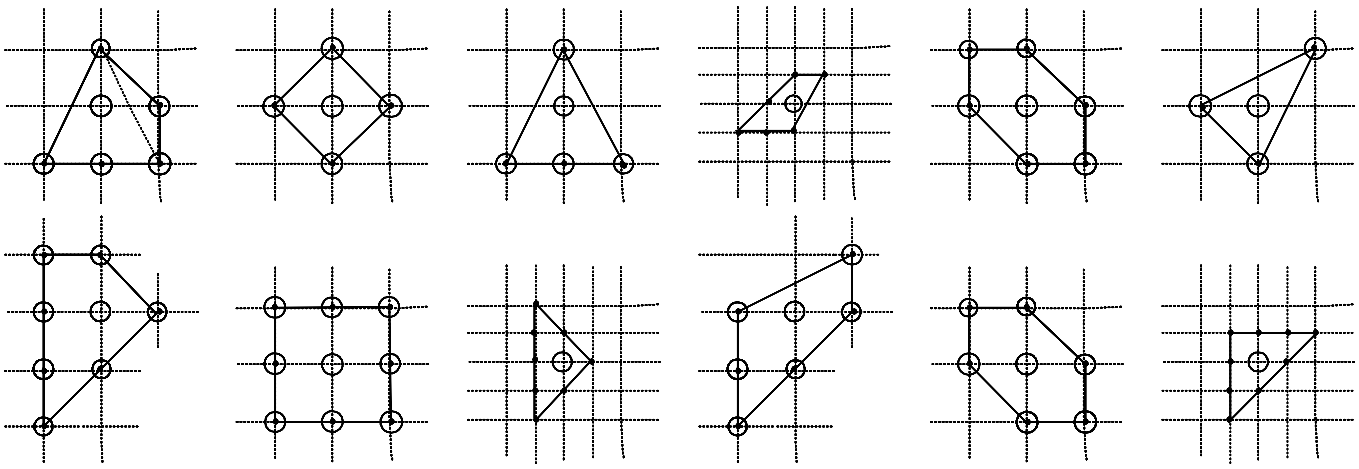
b) The number of solutions of the equation $x^2 + ax + b = 0$ ($x^3 + ax + b = 0$) is equal to the number of tangents to the curve $x^2 = 4y$ ($4x^3 = -27y^2$) passing through the point (a, b).

c) How many solutions has the equation $x^3 + ax + b = 0$ depending on a and b?

d) Try to investigate another equations in such a way.

"12"
Solutions

0. See Figures 6a–d. For the polygon on Figure 2e there is not the dual polygon because it is not convex (see also problem 14). Figure 6e gives an example such that $M = M^*$. Possible observations:
- in the interior of M^* there is exactly one lattice point (see problem 12);
- $M^{**} = -M$ (see problem 11)
- $S(M) + S(M^*) = 6$ which is equivalent to Theorem on 12 since by Pick's theorem (problem 6b) $S(M) = i + \frac{b}{2} - 1 = \frac{b}{2}$.

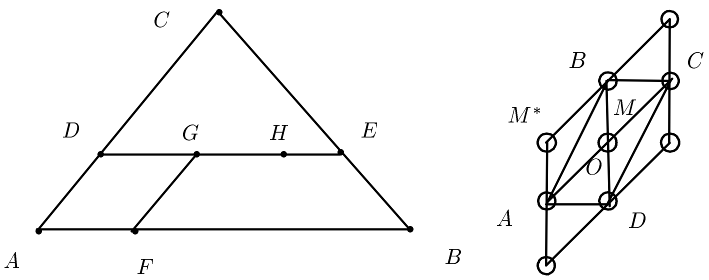


a *b* *c* *d* *e* *f*

Figure 6.

1. Answer: 9. An example is shown on Figure 6f. An idea of construction is suggested by Theorem on 12. As the sum of boundary lattice points of triangles T and T^* is equal to 12 so in order to maximize the number of boundary lattice points of T one should minimize the number of boundary lattice points of T^* and vice versa. Therefore we must take a triangle with only 3 boundary lattice points. Then the dual triangle is required.

Of course the maximality of number 9 follows from Theorem on 12. But we give also an independent proof (Figure 7a). If any side of a triangle contains no more then 2 lattice points (excluding vertices) then there is nothing to prove. Suppose that one side of a triangle, say AB, contains ≥ 3 lattice points (it is possible: see Figure 6b). It is sufficient to prove that any other side contains no more then 1 lattice point. Suppose that AC contains ≥ 2 lattice points. Let D be one of them nearest to A. Draw $DE \parallel AB$, where $E \in AC$. Take on the side AB the lattice point F nearest to A. Let us consider points G and H on the segment DE such that $DG = AF$ and $DH = AF$. Clearly these points are lattice points. Since lattice points divide AB and AC in equal parts, then $CD : CA \geq \frac{2}{3}$ and $AF : AB \leq \frac{1}{4}$. From similarity follows $DE : AB \geq \frac{2}{3}$. Hence, $DH < DE$, i.e., G and H belongs to the interior of the triangle. This contradicts to assumptions. The maximality is proved.



a *b*

Figure 7.

2. Indeed, in this case $O = AC \cap BD$ (Figure 7b) because the point symmetric to the point O with respect to $AC \cap BD$ is a lattice point and belongs to the interior of $ABCD$, so it coincides with O.

It is easy to how that M^* is a parallelogram with sides obtained from the diagonals AC and BD by parallel translations with vectors $\pm\overrightarrow{OB}$ Ë $\pm\overrightarrow{OA}$, respectively.

Since only one lattice point O belongs to these diagonals, then any side of the parallelogram M^* contains one lattice point, hence, $b + b^* = 4 + 8 = 12$.

3.) The triangles ABC and $A'BC$ have the common base BC, and since $AB = BA'$, then the corresponding altitudes are equal. So the areas of these triangles are equal.

b) Let D be the middle point of BC. Note that the lattice is preserved under the symmetry with respect to the point D. Therefore if the triangle ABC is simple, then its image under this symmetry, $A''BC$, also is simple. Similarly, the triangle $A''BA'$ is simple, because it is obtained from $A''BC$ by a symmetry with respect to a point. Hence the interior of the parallelogram $A''A'BC$ does not contain lattice points. In particular, $A'BC$ is simple.

c) Let us consider the minimal rectangle containing our triangle with the sides parallel to lattice lines. Then every side of the rectangle contains a triangle vertex. It is possible only if one of triangle vertices coincides with a rectangle vertex. If we have 3 coinciding vertices, then it is easy to show that our triangle is minimal. If we have 2 coinciding vertices, and they are opposite (for the rectangle), then evidently the third angle of the triangle is obtuse. The other cases are impossible for a simple triangle. Indeed, let the triangle and the rectangle either have a unique common vertex A or they have two common vertices belonging to one side AB of the rectangle. Denote by D the vertex of the rectangle opposite to A, and denote by M the middle point of the side of the triangle opposite to A. Then it is easy to show that the point symmetric to D with respect to M is a lattice point belonging either to the interior of the original triangle or to the interior of its side.

d) It suffices to jump over the vertex B with obtuse angle. Since the side AC is maximal, then $AC > BC$ and $AC > AB = BA'$. Since the angle ABC is obtuse, then it is greater than the angle $A'BC$, hence $AC > A'C$ (because two other sides of the triangles ABC and $A'BC$ are equal).

e) Let us make smaller the length of the maximal side of a simple triangle until it is possible. This process cannot be infinite, because the square of the length of the maximal side is an integer. (by Pythagorean theorem). By d) this process finishes with a minimal triangle.

f) Follows from a) and e).

4. a) Let $ABCD$ be our quadrilateral, $O = AC \cap BD$. Since $ABCD$ does not contain lattice points in the interior and on the sides, it follows that the triangles ABC and ACD are simple. By problem 3d their areas are equal to $1/2$. Hence $BO = OD$. Similarly $AO = OC$, and $ABCD$ is a parallelogram.

b) Without loss of generality the sides of the pentagon $ABCDE$ do not contain lattice points (otherwise we can consider a smaller pentagon). Then by) $ABCD$ is a parallelogram, and $BCDE$ is also a parallelogram. Therefore $A = E$, a contradiction. Hence the required pentagon does not exist.

c) Answer: 6. See the example in Figure. 6e. Let us prove the maximality (it follows evidently from Theorem on 12). Draw a line through the unique lattice point O in the interior of the polygon, that does not pass through the vertices of the polygon. It divides the plane in 2 parts. If the number of the vertices is ≥ 7, then by the Dirichlet principle one of the parts contains at least 4 vertices. Denote them by A, B, C, D. Then the interior of the pentagon $OABCD$ does not contain lattice points, that contradicts to b). The maximality is proved.

5. a) Let O be the unique lattice point in the interior of the triangle ABC. Then the triangles AOB, BOC and COA are simple, hence their areas are equal to $1/2$. Hence O is the barycenter (by a well-known theorem of elementary geometry).

b) By a) O is the mass center of ABC. So $\overrightarrow{B^*C^*} = \overrightarrow{BC} - \overrightarrow{AB} = 3\overrightarrow{BO}$ (A). Since BO is simple, it follows that B^*C^* contains 2 integer points (except vertices). Arguing analogously for A^*B^* and C^*A^*, we obtain $b^* = 9$. So $b + b^* = 12$.

6. a) Answer: $2i + b - 2$. Compute the sum of the angles of all triangulation triangles in two ways.

b) The Pick formula follows from problem 3 and a).

7. The square of the side is an integer and doubled area ia also an integer, so their ratio is rational.