"12"

Problem proposed by S. Dorichenko, V. Prasolov, and M. Skopenkov

introductory problems

Let us consider a convex polygon $M = A_1A_2\dots A_n$ all of whose vertices lie in the lattice of points with integer coordinates (Figure 1 on the left). Suppose that O is the only lattice point in the interior of M. Draw vectors $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \dots \overrightarrow{A_nA_1}$ from the point O and for any obtained segment choose its lattice point nearest to O. Connecting consecutive chosen n points we get a polygon M^*, so called *dual* to the original polygon (Figure 1 on the right).

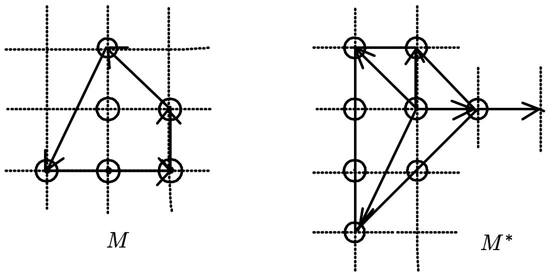

Figure 1.

This series of problems is devoted to a proof of the following theorem discovered only few years ago:

Theorem on 12. *Suppose that O is the only lattice point in the interior of a convex polygon M, the boundary of M contains b lattice points, and the boundary of M^* consists of b^* lattice points. Then*

$$b + b^* = 12.$$

0. Draw the polygons dual to polygons on Figure 2. How many lattice points are in the interior of M^* in each case? What is M^{**}? Is there a relation between areas of M and M^*? Is it possible that $M = M^*$? Formulate your observations and hypothesis and try to prove them.

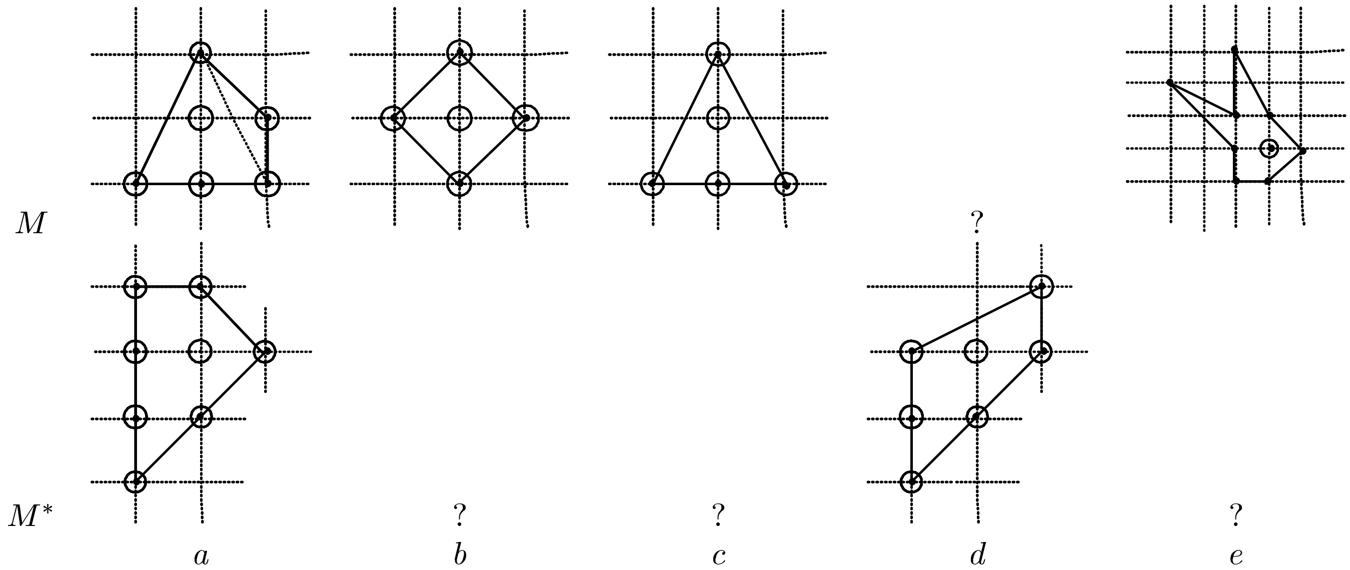

Figure 2.

1. The interior of a triangle contains exactly one lattice point. What is the maximal number of points on its sides?

2. Prove Theorem on 12 for a parallelogram with $b = 4$.

A simple triangle is a triangle without lattice points in the interior and on the boundary (excluding vertices). For a triangle ABC *a jump* is the operation shown in Figure 3: we replace the vertex A by the point symmetric to A with respect to B.

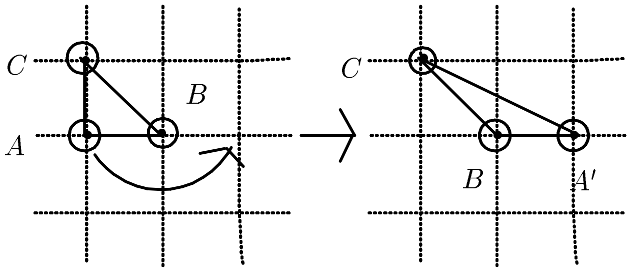

Figure 3.

3. a) The area of a triangle is invariant under jump.

b) For a simple triangle after a jump we obtain a simple triangle.

c) A simple triangle has no more then one obtuse or right angle and the last case is possible only for the triangle having sides 1, 1, $\sqrt{2}$; let us call such a case *minimal*).

d) From any non-minimal triangle one can obtain by a single jump a triangle whose greatest side is less then the greatest side of the original triangle.

e) From any simple triangle one can obtain a minimal triangle by a finite number of jumps.

f) The area of any simple triangle is equal to 1/2.

4. a) If a convex quadrilateral does not contain lattice points in the interior and on the boundary (excluding vertices) then this quadrilateral is a parallelogram.

b) Is there a convex pentagon without lattice points in the interior?

c) In the interior of a convex polygon there is only one lattice point. What is the maximal number of its vertices?

5. a) The boundary of a triangle does not contain lattice points (excluding vertices) and there is only one interior lattice point. Then this point is the centroid.

b) Prove Theorem on 12 for a triangle with $b = 3$.

Let S be the area of a polygon. Suppose that its boundary contains b lattice points and its interior contains i lattice points.

6. a) Let us consider a triangulation of the polygon with vertices at $b+i$ given lattice points. How many triangles may have this triangulation?

b) Prove *Pick's formula:*

$$S = i + \frac{b}{2} - 1.$$

7. Let the ratio of the area of a polygon to the square of one of its sides is irrational (it is true, for example, for equilateral triangle). Prove that one can not draw this polygon (or a polygon similar to it) so that all its vertices are lattice points.

8. The king passed round the chess-board 8×8 squares passing once through any square and returned back with the last move. The broken line connecting centers of squares through which passed the king is not self-intersecting.

a) Find the maximal area bounded by this broken line.

b) Find the maximal length of the broken line.

Deleting a triangle is cutting off from polygon M a simple triangle having two common sides with M. For example, from the polygon on Figure 2a one can obtain the polygon on Figure 2b by deleting a triangle. The reverse operation is called *adding a triangle*.

9. Is it true that every 2 polygons

a) in Figure 2; b) with no lattice points inside; c) which are convex and contain 1 lattice point inside; can be obtained one from another by deleting and adding triangles?

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Basic problems

Let O be the origin of Cartesian coordinate system on the plane. Let us assign to a point $A(a,b) \neq O$ the line A^* given by the equation $ax + by = 1$. This line is called *dual* to the point. And vice versa to any line $ax + by = 1$ assign *dual* point (a,b).

10. Draw configurations dual to configurations on Figure 4. Verify that $l^* \in A^*$ on Figure 4a and lines A^*, B^*, C^* on Figure 4b have a common point. Prove that for any point $C \neq O$ the dual line C^* is orthogonal to OC and goes through the point $C' \in OC$ such that $OC' \cdot OC = 1$.

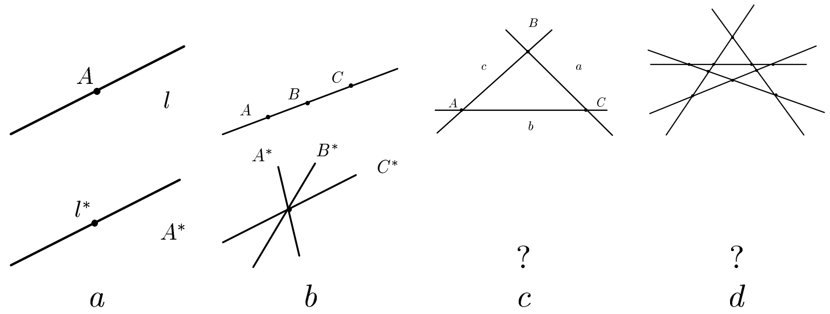

Figure 4.

For a polygon $M = A_1 \dots A_n$ we set $M^* = (A_1A_2)^* \dots (A_nA_1)^*$.

11. Verify that this definition of the dual polygon is compatible with the previous definition (up to a rotation by $90°$). Prove that $M^{**} = M$.

12. If the interior of M contains exactly one lattice point, then the interior of M^* also contains exactly one lattice point.

13. a) Deleting a triangle from M implies adding a triangle to M^*.

b) From any convex polygon with exactly one lattice point in the interior one can obtain a parallelogram with $b = 4$ using deleting/adding triangles.

c) Prove Theorem on 12.

14. Find the figure dual to the non-convex polygon on Figure 2e. Under which assumption on M both definitions of M^* are equivalent? How to generalize Theorem on 12 for non-convex polygons?

Let γ be a curve. The *dual* curve γ^* is the set of all points dual to all tangent lines to γ.

15. Draw the curve dual to the unit circle. Complete Figure 5. Is it true that $\gamma^{**} = \gamma$?

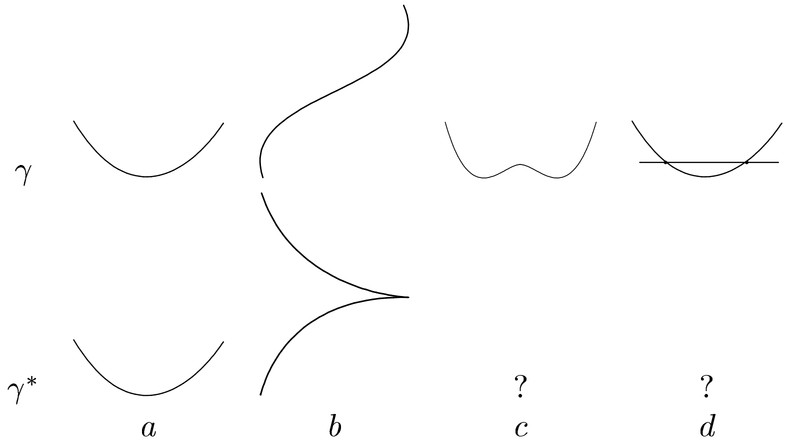

Figure 5.

16. (Open problem.) How to generalize Theorem on 12 for polygons with several lattice points in the interior?

[1] N. B. Vasiliev, *Around the Pick formula*, Kvant **12** (1974). (in Russian)

[2] B. Poonen and F. Rodriges-Villegas, *Lattice polygons and the number 12*, Am. Math. Mon. **107:3** (2000), 238–250.

[3] D. Repovš, M. Skopenkov and M. Cencelj, *An elementary proof of the theorem on 12 integer points*, submitted to Math. Zametki.

[4] S. L. Tabachnikov, *Geometry of equations*, Kvant **10** (1988). (in Russian)

"12"
Supplementary problems

17. One need to draw a broken non self-intersecting line whose vertices are all lattice points in the interior of a $p \times q$ rectangle and whose edges are either vertical or horizontal.

a) Find p and q such that it is possible.

b) Find the length of this line.

c) Find the area bounded by this line.

18. a) Denote by kM a polygon obtained from M by the homothety (dilation) with the coefficient k and the center in the origin. Prove that $2S(M) = n(2M) - 2n(M) + 1$, where $n(M)$ is the number of lattice points in the interior of M and on the boundary.

b) Find and prove an analogous formula for the volume of a polyhedron in space.

19. Find an upper border to the number of boundary lattice points of a convex polygon with n lattice points inside.

An affine transformation is the transformation of the plane given by the formula

$$\begin{cases} x' &= ax + by, \\ y' &= cx + dy, \end{cases}$$

preserving the lattice of integer points (this means that a, b, c, d are integers and $ad - bc = \pm 1$).

20. Classify all the convex polygons

a) without lattice points in the interior; b) with exactly 1 lattice point in the interior; up to an affine transformation.

21. a) Find the image of the line $y = ax + b$ under the transformation $(x, y) \mapsto (\frac{x}{y}, \frac{1}{y})$.

b) The number of solutions of the equation $x^2 + ax + b = 0$ ($x^3 + ax + b = 0$) is equal to the number of tangents to the curve $x^2 = 4y$ ($4x^3 = -27y^2$) passing through the point (a, b).

c) How many solutions has the equation $x^3 + ax + b = 0$ depending on a and b?

d) Try to investigate another equations in such a way.

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Solutions

0. See Figures 6a–d. For the polygon on Figure 2e there is not the dual polygon because it is not convex (see also problem 14). Figure 6e gives an example such that $M = M^*$. Possible observations:

- in the interior of M^* there is exactly one lattice point (see problem 12);
- $M^{**} = -M$ (see problem 11)
- $S(M) + S(M^*) = 6$ which is equivalent to Theorem on 12 since by Pick's theorem (problem 6b) $S(M) = i + \frac{b}{2} - 1 = \frac{b}{2}$.

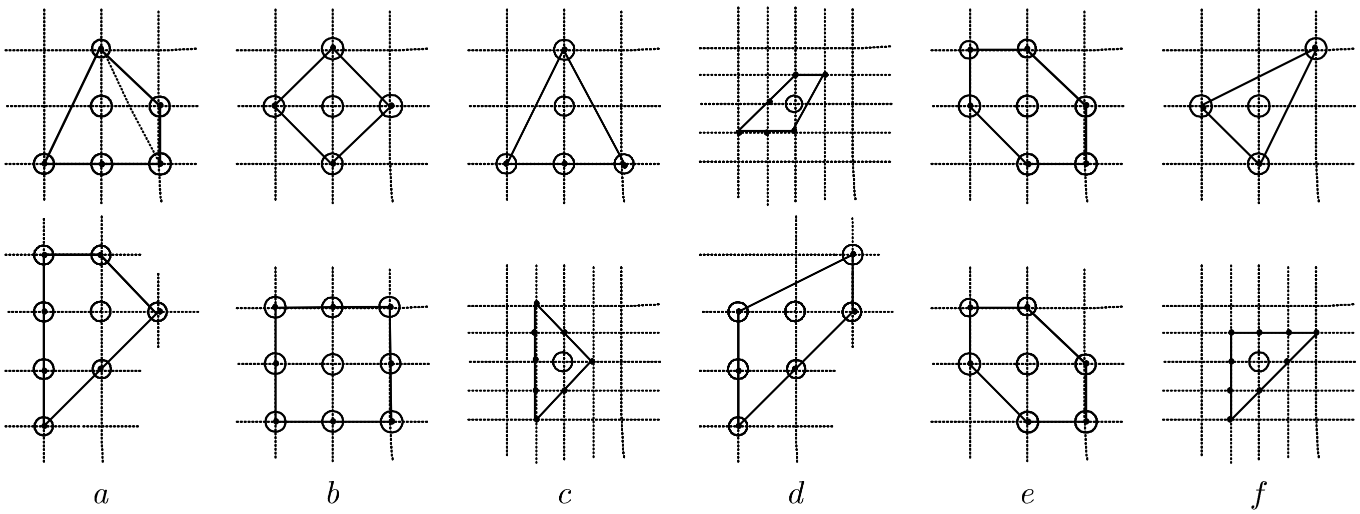

Figure 6.

1. Answer: 9. An example is shown on Figure 6f. An idea of construction is suggested by Theorem on 12. As the sum of boundary lattice points of triangles T and T^* is equal to 12 so in order to maximize the number of boundary lattice points of T one should minimize the number of boundary lattice points of T^* and vice versa. Therefore we must take a triangle with only 3 boundary lattice points. Then the dual triangle is required.

Of course the maximality of number 9 follows from Theorem on 12. But we give also an independent proof (Figure 7a). If any side of a triangle contains no more then 2 lattice points (excluding vertices) then there is nothing to prove. Suppose that one side of a triangle, say AB, contains ≥ 3 lattice points (it is possible: see Figure 6b). It is sufficient to prove that any other side contains no more then 1 lattice point. Suppose that AC contains ≥ 2 lattice points. Let D be one of them nearest to A. Draw $DE \parallel AB$, where $E \in AC$. Take on the side AB the lattice point F nearest to A. Let us consider points G and H on the segment DE such that $DG = AF$ and $DH = AF$. Clearly these points are lattice points. Since lattice points divide AB and AC in equal parts, then $CD : CA \geq \frac{2}{3}$ and $AF : AB \leq \frac{1}{4}$. From similarity follows $DE : AB \geq \frac{2}{3}$. Hence, $DH < DE$, i.e., G and H belongs to the interior of the triangle. This contradicts to assumptions. The maximality is proved.

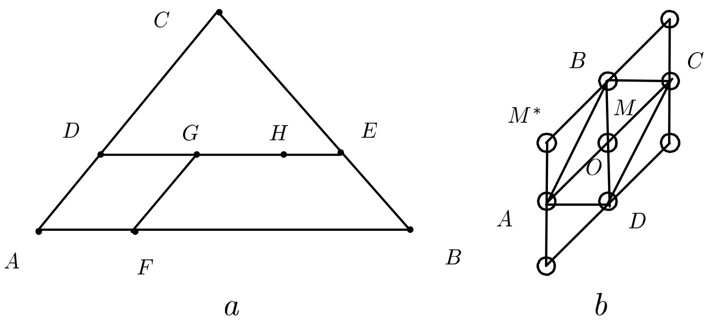

Figure 7.

2. Indeed, in this case $O = AC \cap BD$ (Figure 7b) because the point symmetric to the point O with respect to $AC \cap BD$ is a lattice point and belongs to the interior of $ABCD$, so it coincides with O.

It is easy to how that M^* is a parallelogram with sides obtained from the diagonals AC and BD by parallel translations with vectors $\pm\overrightarrow{OB}$ p $\pm\overrightarrow{OA}$, respectively.

Since only one lattice point O belongs to these diagonals, then any side of the parallelogram M^* contains one lattice point, hence, $b + b^* = 4 + 8 = 12$.

3.) The triangles ABC and $A'BC$ have the common base BC, and since $AB = BA'$, then the corresponding altitudes are equal. So the areas of these triangles are equal.

b) Let D be the middle point of BC. Note that the lattice is preserved under the symmetry with respect to the point D. Therefore if the triangle ABC is simple, then its image under this symmetry, $A''BC$, also is simple. Similarly, the triangle $A''BA'$ is simple, because it is obtained from $A''BC$ by a symmetry with respect to a point. Hence the interior of the parallelogram $A''A'BC$ does not contain lattice points. In particular, $A'BC$ is simple.

c) Let us consider the minimal rectangle containing our triangle with the sides parallel to lattice lines. Then every side of the rectangle contains a triangle vertex. It is possible only if one of triangle vertices coincides with a rectangle vertex. If we have 3 coinciding vertices, then it is easy to show that our triangle is minimal. If we have 2 coinciding vertices, and they are opposite (for the rectangle), then evidently the third angle of the triangle is obtuse. The other cases are impossible for a simple triangle. Indeed, let the triangle and the rectangle either have a unique common vertex A or they have two common vertices belonging to one side AB of the rectangle. Denote by D the vertex of the rectangle opposite to A, and denote by M the middle point of the side of the triangle opposite to A. Then it is easy to show that the point symmetric to D with respect to M is a lattice point belonging either to the interior of the original triangle or to the interior of its side.

d) It suffices to jump over the vertex B with obtuse angle. Since the side AC is maximal, then $AC > BC$ and $AC > AB = BA'$. Since the angle ABC is obtuse, then it is greater than the angle $A'BC$, hence $AC > A'C$ (because two other sides of the triangles ABC and $A'BC$ are equal).

e) Let us make smaller the length of the maximal side of a simple triangle until it is possible. This process cannot be infinite, because the square of the length of the maximal side is an integer. (by Pythagorean theorem). By d) this process finishes with a minimal triangle.

f) Follows from a) and e).

4. a) Let $ABCD$ be our quadrilateral, $O = AC \cap BD$. Since $ABCD$ does not contain lattice points in the interior and on the sides, it follows that the triangles ABC and ACD are simple. By problem 3d their areas are equal to $1/2$. Hence $BO = OD$. Similarly $AO = OC$, and $ABCD$ is a parallelogram.

b) Without loss of generality the sides of the pentagon $ABCDE$ do not contain lattice points (otherwise we can consider a smaller pentagon). Then by) $ABCD$ is a parallelogram, and $BCDE$ is also a parallelogram. Therefore $A = E$, a contradiction. Hence the required pentagon does not exist.

c) Answer: 6. See the example in Figure. 6e. Let us prove the maximality (it follows evidently from Theorem on 12). Draw a line through the unique lattice point O in the interior of the polygon, that does not pass through the vertices of the polygon. It divides the plane in 2 parts. If the number of the vertices is ≥ 7, then by the Dirichlet principle one of the parts contains at least 4 vertices. Denote them by A, B, C, D. Then the interior of the pentagon $OABCD$ does not contain lattice points, that contradicts to b). The maximality is proved.

5. a) Let O be the unique lattice point in the interior of the triangle ABC. Then the triangles AOB, BOC and COA are simple, hence their areas are equal to $1/2$. Hence O is the barycenter (by a well-known theorem of elementary geometry).

b) By a) O is the mass center of ABC. So $\overrightarrow{B^*C^*} = \overrightarrow{BC} - \overrightarrow{AB} = 3\overrightarrow{BO}$ (A). Since BO is simple, it follows that B^*C^* contains 2 integer points (except vertices). Arguing analogously for A^*B^* and C^*A^*, we obtain $b^* = 9$. So $b + b^* = 12$.

6. a) Answer: $2i + b - 2$. Compute the sum of the angles of all triangulation triangles in two ways.

b) The Pick formula follows from problem 3 and).

7. The square of the side is an integer and doubled area is also an integer, so their ratio is rational.

8. a) Answer: 31 (follows from the Pick formula).

b) Answer: $28+36\sqrt{2}$. It is easy to find the path of the king such that 36 of 64 moves have length $\sqrt{2}$ (in diagonal direction). Let us prove that the number 36 is maximal. For every segment of length $\sqrt{2}$, belonging to king's path let us construct the square 1×1 having this segment as a diagonal. One half of the square is outside the polygon bounded by king's path. But the whole area, covered by these halves, is not greater than 49-31=18, because they all belong to the square 7×7 (see a)). So the number of diagonal moves is not greater than 36.

9. The answer is "yes" for all three cases. Problem) follows from c).

b) It is sufficient to prove that from a given polygon without interior lattice points we can obtain a minimal simple triangle by deleting and adding triangles (see problem 3c). Take a triangulation of the polygon and delete triangles one by step. We get a simple triangle. Now the problem follows from 3e, because the jump of a simple triangle is the result of adding a triangle followed by deleting the initial triangle.

c) Let us prove that from any parallelogram with $b=4$ and $i=1$ we can obtain the parallelogram in Figure 2b by deleting/adding triangles. Then our problem follows from 13b. Now it is sufficient to apply problem 3e to the triangle AOB.

10. See Figure 8. Let us prove that pictures in Figure 8ab are correct. Let the point A in Figure 8 has coordinates (u,v), and the line l is given by the equation $ax+by=1$. Then $A\in l \Leftrightarrow au+bv=1 \Leftrightarrow l^*\in A^*$, as required. Further, let l be the line through the points in Figure 8b. Then from previous argument it follows that three lines A^*, B^* and C^* have a common point l^*.

Now let us prove the last assertion of problem 10. Suppose that the point C has coordinates (a,b), then $ax+by=1$ is the equation of the line C^*. It easy to show that $C^*\perp OC$. It remains to prove that $\rho(O,C^*)\cdot OC=1$, that is *the distance ρ from the line $ax+by=1$ to the origin is equal to $\frac{1}{\sqrt{a^2+b^2}}$*. For this purpose it is sufficient to compute the area of the triangle XOY, where X and Y are the intersection points of our line and the axes Ox and Oy respectively, in two ways.

One can see that the line dual to A is the *polar* to this point with respect to the unit circle.

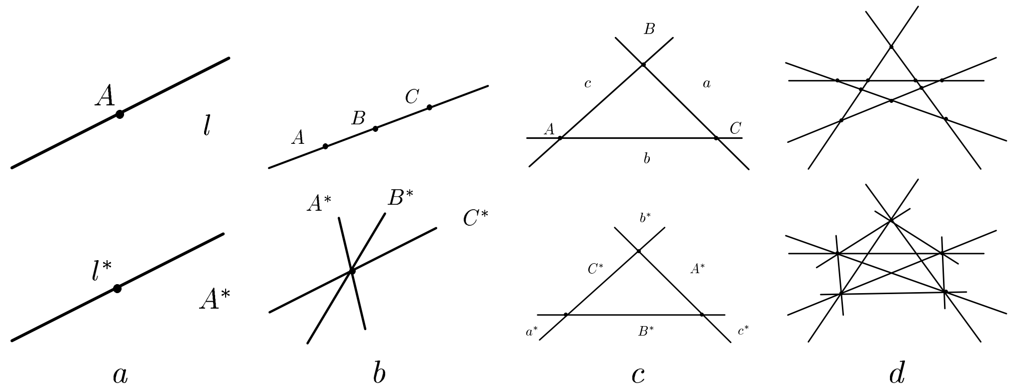

Figure 8.

11. The proof is based on the following remark, that follows from problem 3f:
O is the unique lattice point in the interior of M $\Leftrightarrow$ for any two neighbor lattice points A and B on the boundary of M the area of the triangle AOB is equal to $1/2$.

To prove the equivalence of two definitions of the dual polygon it is sufficient to verify that the vector $\overrightarrow{OC}$, where $C=(AB)^*$, is obtained from $\overrightarrow{AB}$ by the rotation at the angle $90°$ in counterclockwise direction. By problem 10 $AB\perp OC$. By the same problem $\rho(O,AB)\cdot OC=1$. On the other hand by the above remark $\frac{1}{2}=S(AOB)=\frac{1}{2}\rho(O,AB)\cdot AB$. Hence $AB=OC$. So we have proved that OC is obtained from AB by the rotation at the angle $90°$. Evidently, if the boundary of M is oriented clockwise, then this rotation is counterclockwise.

From the second definition of M^* the equality $M=M^{**}$ is obvious.

12. Suppose that the interior of M^* contains another lattice point Q distinct from O. Then Q belongs to the interior of a triangle AOB, where A and B are neighbor vertices of the polygon M. The triangles OQA and OQB are simple, so by problem 3f their areas are equal to $1/2$. Therefore Q belongs to the median OM. By definition of M^* on the boundary of M there are three lattice points

D, E and F such that $\overrightarrow{DE} = \overrightarrow{OA}$ and $\overrightarrow{EF} = \overrightarrow{OB}$. Evidently, $DF \parallel OM$, hence, $DF \parallel OQ$. But $DF = 2OM > 2OQ$, therefore the interior of DF contains at least 2 lattice points. Since M is convex, then we have to possibilities: either DF belongs to the interior of M or DF is a part of boundary of M. In the first case we have that the interior of M contains at least two lattice points. In the second case $M = DEF$ does not contain lattice points at all. Contradiction. Therefore such a lattice point Q does not exist.

13. a) It is sufficient to prove that deleting a simple triangle, say $A_1A_2A_3$, from M gives adding a simple triangle $A_{12}A_{13}A_{23}$ to M^* (Figure 9). Here by A_{kl} we denote the point such that $\overrightarrow{OA_{kl}} = \overrightarrow{A_kA_l}$. This assertion follows easily from the second definition of M^* and problem 4a.

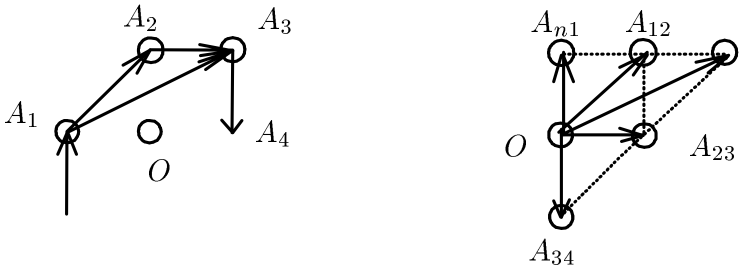

Figure 9.

b) Indeed, suppose at first that M has a diagonal not passing through O. Cut M along this diagonal and consider the obtained part not containing O. This part necessarily contains a simple triangle of the form $A_{i-1}A_iA_{i+1}$. Deleting it we decrease the number b. Repeat this operation until it is possible. Repetition is impossible only in the following 3 cases (when such a diagonal does not exist):
A) $b = 4$, $M = ABCD$, $O = AC \cap BD$. Since the segments OA, OB, OC and OD do not contain lattice points, then $OA = OC$ and $OB = OD$, that is $ABCD$ is the required parallelogram.
B) $b = 4$, $M = ABD$, and C belongs to the segment BD. In this case let us denote by D' the point symmetric to D with respect to O, and denote by E the middle point of $D'B$. The required sequence of deleting/adding of triangles has the form:

$$ABCD \to AEBCD \to AD'EBCD \to AD'ECD \to AD'CD \quad \text{(Figure 10 on the left).}$$

C) $b = 3$, $M = ABC$. In this case denote by A' and C' the points symmetric to A and C, respectively, with respect to O. The required sequence of deleting/adding of triangles has the form:

$$ABC \to AC'BC \to AC'BA'C \to AC'A'C \quad \text{(Figure 10 on the right).}$$

So we obtained the required parallelogram in every case.

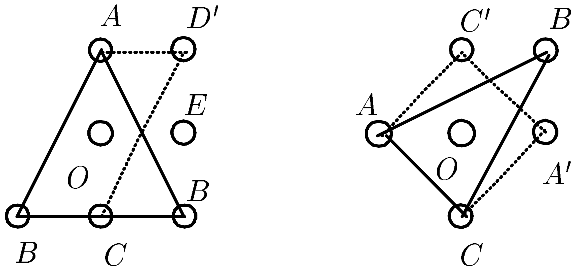

Figure 10.

c) follows from a), b) and problem 2.

14. One can see that the figure dual to a not convex polygon is a broken line, which may have self-intersections. So it is natural to generalize Theorem on 12 to broken lines, possibly, with self-intersections. A segment AB of a broken line is called *positive*, if when the point X moves along this segment from A to B the segment OX rotates clockwise. Otherwise the segment AB is *negative*. This notion is well-defined, because the area of the triangle AOB is equal to $1/2$, in particular, the points A, O and B are not collinear. Denote by b_+ and b_- the number of positive and negative segments of the broken line respectively, and by b_+^* and b_-^* the similar numbers for the dual broken line.

Theorem on 12 for broken lines. *Let M be a closed broken line with vertices at lattice points (and all the lattice points of M we consider as vertices, possibly, with the angle $180°$). Suppose that there is a lattice point O such that for any segment AB of the broken line M the area of the triangle AOB is equal to $1/2$. Suppose that M rotates r times around the point O. Then*

$$b_+ - b_- + b_+^* - b_-^* = 12r.$$

This theorem is proved in the article [2]. An elementary proof is unknown.

The two above definitions of the dual polygon are equivalent if and only if all the sides of M are positive and for any segment AB of the bondary of M the area of the triangle AOB is equal to $1/2$. It is proved analogously to the solution of problem 11. Note that for the broken lines M satisfying the assumptions of Theorem on 12 for broken lines there is an easy construction of M^* like the first definition of the dual polygon.

15. The curve dual to the unit circle is this unit circle itself (it follows from problem 10). Figure 11 completes Figure 5.

Figure 11.

Indeed, for any smooth curve γ we have $\gamma^{**} = \gamma$. Let us give a sketch of the proof. Consider the curve γ^*. To construct the dual curve one should consider the tangent lines to γ^*. Choose a pair of close points in γ^* and draw a line through them instead of a tangent line (Figure 11d on the top). The dual picture is given in Figure 11d on the bottom. If we make the chosen points in γ^* closer to each other, then the drawn line approaches to a tangent line, and the point dual to this line approaches to a point in the curve γ. So the set of tangent lines to γ^* gives the initial curve γ, that is $\gamma^{**} = \gamma$.

Many other interesting examples are given in the article [4].

16. It is an open problem.

17. Answer: a) for $p, q \geq 3$ and $(p-1)(q-1)$ even; b) $(p-1)(q-1)$; c) $\frac{(p-1)(q-1)}{2} - 1$. The solution follows easily from Pick's formula.

18. a) follows immediately from Pick's formula, and it can be also proved analogously to this formula.

b) There are many distinct formulas, generalizing a), e. g.

$$6V(M) = n(3M) - 3n(2M) + 3n(M) - 1 \qquad \text{or} \qquad 6V(M) = n(2M) - 2n(M) - b(M) + 3.$$

This formulas are proved in an interesting paper [5]. In this article it also shown how to obtain all such formulas. See also an exciting paper [6] of the same author.

19. Denote by $n(i)$ the maximal number of the vertices of a convex polygon with exactly i lattice points in the interior. Let us give the list of $n(i)$ for some small i:

i	0	1	2	3	4	5	...
$n(i)$	4	6	6	6	8	7	...

The equalities $n(0) = 4$ and $n(1) = 6$ were proved in fact in problem 4bc. To prove the equality $n(2) = 6$ it suffices to draw a line through the two internal lattice points and then argue as in the solution of 4c. For larger i the jury knows the following estimate for the number $n(i)$:

$$n(i) \leq 2i \text{ for } i \geq 3.$$

To prove the estimate consider the convex hull of i internal lattice points. This is either a segment or a polygon with not more than i vertices. In the first case, arguing as in the solution of problem 4c, it is easy to prove that $n(i) \leq 6$. In the second case take an arbitrary point O (not necessarily lattice point) in the interior of the convex hull and draw the rays beginning at the point O through all the vertices of the convex hull. Without loss of generality we may assume that the vertices of the initial polygon do not belong to these rays. These rays divide the plane into not more than i parts. If the number of vertices $n(i) > 2i$, then by the Dirichlet principle one of these parts contains at least 3 vertices. Together with the two vertices of the convex hull belonging to the same part they constitute a convex pentagon without lattice points in the interior, that is a contradiction with problem 4b. The required estimate is proved.

Notice that $n(5) < n(4)$ (!). It can be proved that $n(i) < Cn^{1/3}$ for some constant C, but the proof is rather complicated.

20. a) The classification of convex polygons without lattice points in the interior is given in [7].

b) The classification of convex polygons with exactly 1 lattice point in the interior is given without proof in the article [2]. It turns out that there are only 16 such polygons up to an affine transformation. The proof is based on the same idea as in a), see the paper [7].

21. a) Answer: the line $ax + by = 1$

b) The number of the solution of the equation $x^2 + ax + b = 0$ is equal to the number of the solutions of the system

$$\begin{cases} y = -x^2, \\ y = ax + b. \end{cases}$$

After the transformation $(x, y) \mapsto (\frac{x}{y}, \frac{1}{y})$ this system is reduced to

$$\begin{cases} y = -x^2, \\ ax + by = 1. \end{cases}$$

The number of the solutions of the obtained system is equal to the number of intersection points of the curve $y = -x^2$ and the line $ax + by = 1$. But it is exactly the same as the number of tangents from the point (a, b) to the dual curve (Figure 11d). It remains to note that the curve $x^2 = 4y$ is dual to $x^2 = -y$ (it can be verified by differentiation). For the cubic equation the argument is similar.

c) Answer: $\begin{cases} 3, & \text{if } D < 0, \\ 2, & \text{if } D = 0 \text{ and } ab \neq 0, \\ 1, & \text{if } D > 0 \text{ or } a = b = 0, \end{cases}$

where $D = 4a^3 + 27b^2$ is the discriminant of the cubic equation. The solution is clear from Figue 11b. One can find the details about dual curves and their relation to solving equation in article [4]. It is interesting that if we study the systems of algebraic equations then there naturally appear lattice polygons (the Newton polygons). Two easy papers on the Newton polygons are [5] and [6] in the above.

[5] A. Kushnirenko, *Lattice points in polygons and polyhedra*, Kvant **4** (1977). (in Russian)
[6] A. Kushnirenko, *The Newton polygons*, Kvant **6** (1977). (in Russian)
[7] A. G. Khovansky, *Newton polygons, curves on toric varietes and the reversion of the Veil theorem*, Uspekhi Mat. Nauk **52:6** (1997), 113–142. (in Russian)