# The automatic machines and the finitely presented semigroups.

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8 августа 2004 г.

## 1 Introduction.

The following project aims to construct finitely given systems with the complicated structure and exotic properties. It may be intersting because of the traditional high algebra questions conserning the structure of finetely presented algebraic systems like groups, semigroups, rings. For example, it is important to discribe the conditions for a nilring to be nilpotent. The same question about semigroups is also intresting. In order to reach semigroups you need to study some definitions and examples.

There are three sections in this project. Every section starts with easy problems. They are necessary to develop the ideas of the main problems. The last problem of each section is considered to be primary. The aim is to construct a finitely presented non nilpotent nilsemigroup (Section C.) It may be useful to study automata with finite memory (section A) in order to turn to semigroups. Section B is good to use some techniques in finitely presented semigroups. The primary result of this section (construction of finitely presented semigroup with nonintegral Gelfand-Kirillov dimension) is intresting itself.

## 2 Section A. The automatic machines and common strategies.

In a first problems of this section we will examine a standard finite automatic machine: it is a robot moving on a cellular plane. The robot can do some actions defined by his program. These actions are:

1. Move forward (to the next cell);
2. Rotate in the current cell;
3. Drop a pebble into the current cell (if the robot has a pebble);
4. Look for a pebble in the current cell;
5. Pick up a pebble from the current cell.

Every problem defines how many pebbles the robot has. The program consists of numerated list with instuctions for the robot. Some of these instructions are jumps to another instructions (like GOTO operator in computer languages). We will discuss the opportunities of walking through some sets of cells. By *walking through* or *investigating* the set of cells we mean that the robot ruled by his program can walk through *every* cell in this set.

First of all we should examine some of the robot's skills

**Example.** Let us prove that the robot with two pebbles can investigate the tape - an infinite strip with width 1. Let the robot be in cell number 0 with two pebbles. Let us write the robot's program.

1. Drop a pebble.
2. Go to cell number 1.
3. Drop a pebble.

Now we are going to organize a shuttle-moving of the robot between the pebbles. Besides, the robot will increase the distance between the pebbles by "pushing" them. We can obtain this by using the program below:

4. Turn around. (turning at 180 degrees)
5. Move forward.
6. Look for a pebble.

7. If there is no pebble – go to 5.
8. Pick up the pebble and move forward.
9. Drop pebble and go to 4.

It is easy to see that the robot will investigate all the cells on the tape.

Could the robot without any pebbles investigate the tape? The answer is "no". In order to prove this we should formalise the construction of the robot. It is easy to see that every robot's action depends on the number of the current instruction and the presence of a pebble in the current cell. Let us call these factors by *the robot's inner condition.* Obviously, there is a finite number inner conditions of the robot. In most cases, the robot changes his inner condition after an action. According to the Dirihlet Principle, some inner condition of the robot will happen again. So, let it take $t$ seconds. The robot has moved right for $a$ cells. It is easy to see that in another $t$ seconds the robot will also move right for another $a$ cells. It is clear that the robot cannot reach some cells placed to the left of his starting position.

**A0.** *Prove that the robot with one pebble cannot investigate the tape.*

**Hint.** You should consider two cases: 1) the robot does not go away from his pebble at the distance more then some $N$;

2) the robot goes away from his pebble at any distance.

**A1.** *Prove that the robot with* 4 *pebbles can investigate the plane.*

**A2.** *Prove that the robot with* 3 *pebbles can investigate the plane. Could the robot with* 3 *pebbles investigate the three dimensional space?*

Let some borders of cells be impassable. The robot can see these barriers between the cells, when he is situated in one of these cells.

**A3.** *The border of the halfplane is impassable line. Prove that the robot with* 1 *pebble can investigate the halfplane.*

**A4.** *Prove that the robot with* 1 *pebble cannot investigate the plane with four cuts. (pic 1.) Prove that the robot with* 2 *pebbles can do this.*

The robot is moving on the vertices of the graph. This graph may be infinite.

**A5.** a) *The graph is called tree if there are no cycles. Let tree be infinite. Suppose any vertice of the tree has degree equal* 3. *Prove that the robot with* 1 *pebble cannot investigate this tree.*

b) *The same question for the robot with* 2 *pebbles.*

**A6.** *Prove that the robot with* 2 *pebbles cannot investigate the plane.*

**Hint.** Use the ideas of the problems A0 and A4.

The robot without any pebbles is moving on the first quarter of the plane. The borders are impassable walls. The robot can see a wall then he is in the cell with this wall.

**A7.** *Let the robot be in the cell with coordinates* $(2^n, 0)$. *Write a universal (not depending on* $n$*) program, for moving the robot into the cell* $(3^n, 0)$ *with stopping in that cell.*

Let us call the transition above as *transition from* $2^n$ *to* $3^n$.

**A8.** *Write the programs for the robot to make transitions below:*

*a) from* $2^n$ *to* $6^n$;
*b) from* $2^n$ *to* $2^{2^n}$;
*c) from* $2^n * 3^m$ *to* $2^n * 3^m * 5^{mn}$;
*d) from* $2^n * 3^m$ *to* $2^n * 3^m * 5^{m+n}$;
*e) from* $2^n$ *to* $2^{[\sqrt{n}]}$;
*f) from* $2^n$ *to* $2^{n^2}$;
*g) from* $2^n$ *to* $2^{k_n}$, the $k_n$ is the $n$-th decimal digit of the $\sqrt{2}$;
*h) from* $2^n$ $\varkappa$ $2^{k_n}$, the $k_n$ is the $n$-th decimal digit of the $\pi$;

**A9.** *Prove that the robot with* 3 *pebbles can investigate n-dimensional space.*

**A10.** *Prove that the robot with* 3 *pebbles can investigate a tree with degree of each vertice equal to* 3.

**Summury.** *Summarize the results above in the table. (Investigating the tape, the plane, the halfplane, the quarter, the space, the tree) with* 0, 1, 2, 3 *pebbles.*

Now we are going to discuss a set of finite automatic machines.

**Common problem.**

There are $N$ soldiers in the formation. Every soldier has a finite amount of memory and can talk to his two neighbours only. Two soldiers on the edges of the formation have only one neighbour. One of soldiers on the edge gets an order. The main task of the soldiers is to fire simultaneously in some time.

**A11.** *Let* $N = 4$. *Every soldier can push his neighbours with his elbows only. Can one state the rules for the soldiers, allowing them to solve the task – fire simutaneously?*

**A12.** *Let* $N = 5$, *and the soldiers are no more pushing with elbows each other. Every soldier can say some messages to his neighbors. There are two posible messages: a and b. Besides, every soldier can remember one integer from* 1 *to* 16. *So, every soldier knows the instructions we gave him and every current moment he remembers one integer from* 1 *to* 16. *These are some possible instructions for the soldiers:*

*if you have the number* 6 *– tell "a" to the right and remember new number* 8

*or*

*if you recieve "b" from the left + and you have* 7 *– tell "a" to the left and remember* 5.

*Can one state a finite list of instruction for the soldiers allowing them to solve the task?*

We have discussed some examples of $N$ with some conditions for the number of possible signals. Now let $N$ be large enough and unknown. An amount of memory and the list of possible signals are bounded.

**A13.** *(Mayhill strikers problem). There are* $N$ *soldiers in the formation. Every soldier has a finite amount of memory and can remember a finite list of words. Every soldier can communicate to his neighbors. Every second some of the soldiers can say some words from the list to their neighbours (it is called send a signal). One of the soldiers on the edge gets the order. The main task of the soldiers is to fire simutaneously after some amount of time. Can one state a finite list of instructions for the soldiers if* $N$ *is unknown?*

In order to deal with this problem we need some formalization. It is good to consider the soldiers as a finite automatic machines. These machines contain $k$ storages of memory. Every storage can contain 0 or 1. Every automatic machine can send or recieve $p$ different signals. Machines can simateniosly send or receive left and right signals. The instructions for the automatic machines are the list of rules, defining how automatic machine changes its memory storages depending on the received signals and the old memory storings. Besides, the rules define how automatic machines send signals depending on the received signals and the memory storings. Let us call the current condition of memory as *inner automatic machine condition.*

**Hint.** We can organaize two signals: the first is spread further as soon as it is received (the next second) and the other - with a delay of two seconds. So, the first signal will move three times faster than the second. the last soldier sends the first signal back when he receives it. You should find the place owhere the signals will meet and define some additional instructions for the soldiers for this case (receiving two signals at once).

**A14.** *Can one organize the soldiers so that they would fire simultaneously in* $2N - 2$ *seconds?*

## 3 Section B. The finitely presented semigroups.

Let us state some definitions.

**Semigroups language.** Let $A = \{a, b, c, \ldots\}$ be the finite alphabet. Let *the word* be a finite letter consistency, for example, *abbc*, *a*, *cabcabcbbb*, etc. The rusult of attaching the word $B$ to word $A$ is called *the product of* $A$ *and* $B$ and is denoted by $A * B$. So the alphabet $A$ defines the set of words with product operation.

he product of some two elements is the element of the set. The set of elements $S$ under some product operation $*$ is called *a semigroup* if the following condition holds for any $a, b, c \in S$: $a * (b * c) = (a * b) * c$.

This condition is called *associative condition.*

**Examples.**

The set of integer with addition is a semigroup. The set of rational numbers form a semigroup under multiplication. The set of rational numbers under division is not a semigroup.

The set of words defined by the alphabet is a semigroup under the word attaching operation. (It is easy to see that the associative condition holds.) Further, this set of words is called *a free semigroup.* The semigroup is said to be *a finitely generated* if the alphabet is finite. We are going to consider the finitely generated semigroups only.

The $Y$ word is said to be the $N-thpower$ of $X$ if $Y = X * X * \ldots * X$ ($N$ times).

**Problem.** Prove that $U * V = V * U$ if and only if there exists a word $A$ such that $X = A^n$ and $Y = A^n$.

**Defining relations.**

Suppose we can change one subword to another. For example, $ab$ we can change to $bca$. Then the words $ab$ and $bca$ is said to be *"equivalent"* or *"equal"*. So if the word could be transfered to the another word by changing some subword then these words are equivalent. For example, if $ab = bca$, then $aaab = aabca = abcaca = bcacaca$, and we can conclude that $aaab$ and $bcacaca$ are equal words.

Since, we can get *equivalence classes.* These classes are the sets of pairwise equal words. These classes are the elements of a new constructed semigroup. If we want to find the product of the two classes, we should take two representatives-words, then find their product. This word is the representative of some class. This class is said to be the product of our two classes. It is easy to see that the result does not depend on representatives.

In reality we are considering words as the elements of a semigroup. We always know that some of them are equal.

What should we do to define a semigroup? In order to construct a semigroup we write some *defining relations.* These relations are the pairs of equal words. These word are said to be equal by definition. The two words $A$ and $B$ are called equal, if they can be obtained from the each other by some changes of the equal subwords.

**Example.** Let us consider the construction of the semigroup with the defining relations: $ba = abb$; $aa = a$.

Take a word $X$. If the word $aa$ is the subword of the $X$, we should apply the second relation. Hence, it is easy to see that the word $X$ has the form:

$$b^{k_1}ab^{k_2}a\ldots ab^{k_n}, \qquad where k_1 \geq 0, k_n \geq 0, k_2, k_3, \ldots k_{n-1} > 0.$$

On the other hand, using the first relation we can "transfer" any $a$ letter to the right. So we obtain that the word has we form $a^nb^k$. Now we apply the second relation again, transfering the word to the form $ab^n$. So any word containing letter $a$ has the form $ab^n$. Hence we can assume that this semigroup can be described as the set of pairs $(n, k)$, where $n = 0$ or $1$, $k \geq 0$. It is easy to see that the product of pairs is satisfying the following relations: $(n, k_1) * (0, k_2) = (n, k_1 + k_2)$, $(n, k_1) * (1, k_2) = (1, 2k_1 + k_2)$.

A semigroup with the finite set of the defining relations is called *finitely presented.* We are going to consider only finitely presented semigroups. The element of the semigroup is called null and is denoted 0 if for any element $x$ we get $0 * x = x * 0 = 0$. It is easy to prove that if a null exists then there exist a unique null. A semigroup is called /it finite if the set of the elements is finite.

We are using the defining relations to state the rules for words transformations. The semigroups and the automatic machines are the relative objects. Automatic machines handle some data and make the actions depends on the recieved information. Semigroups handle the words by defining relations. For example, it is possible to null some words using some special relations.

**Example.**

Let us construct the finitely presented semigroup $S$ such that the following condition holds: all the words not equal to null have the form $a^nb^kc^l$, where $n, k, l \geq 0$. In this case the alphabet contains three letters $a, b, c$. Let us state the defining relations $ba = 0, ca = 0, cb = 0$. It is easy to see, that if the word contains $c$, then $c$ is in the end of the word (because of $ca = 0, cb = 0$.) Note that the end of the word may contains several $c$ i.e. $c^l$, $l \geq 0$. Similarly, using $ba = 0$ we obtain that only $c$ can be to the right of $b$. Hence we obtain the given form.

In the Mayhill problem the soldiers communicated by sending the signals to each other. In the semigroup the special letter can be the signal if it is moving through the word. The moving is possible due to the defining relations.

> **Example.** Let $L$ be the finite alphabet and let $A, B$ be the letters of $L$. Let us construct the finitely presented semigroup such that the following condition holds: every word not equal to null containg both $A$ and $B$.
>
> Let alphabet $L$ be $A, B, c, d, e, t$. Let us state the relations $A = At$, $A = tA$, $tc = ct$, $td = dt$, $te = et$, $tB = 0$, $Bt = 0$.
>
> Suppose that the word $W$ contains the letters $A$ and $B$. Let us prove that $W = 0$. Let $A$ occur to $W$ to the left of $B$. Hence $W$ contains the subword $AXB$, where the word $X$ consists of the letters $c, d, e, t$. Hence we can conclude $AXB = AtXB = AXtB = 0$.

**B0.** *Do the following sets form a semigroup?*
*a) The negative numbers under multiplication;*
*b) The integer numbers under summation of square;*
*c) The squares on the plane under intersection.*
*d) The words in the alphabet $\{a, b\}$ with the first letter equal to $a$ under attaching;*
*e) The numbers have a form $n^3 + m^3$ (where $m, n$ are natural numbers) under multiplication.*
*f) The set of the space motions saving the unite cube under superposition.*

**B1.** *Describe the semigroup presented by these relations: $bab = 0$; $baa = baca$; $ca = ac$; $cbb = 0$; $bbb = 0$. Is it finite?*

**B2.** *If the semigroup is finite there exists an element $a$ such that $a * a = a$. Prove it.*

The element $a$ of the semigroup is called *idempotent* if $a^2 = a$.

**B3.** *There are finite number of the letters in the alphabet of the indians. Suppose that any word written twice is initial word. The replacing of any subword by its square does not change the whole word. For example,* **cowbwboy** $=$ **cowboy**. *Prove that there are finite number of words in the alphabet of the indians if:*
*a) $N = 2$;*
*b) $N = 3$;*
*c) $N$ is an arbitrary natural number.*

**Addition.** This statement has the form: *Every finitely generated idempotent semigroup is finite.*

**B4.** *Let $A, B, C \in L$ Construct the finitely presented semigroup satisfying the following condition: every word containing all three letters $A, B, C$ is equal to null.*

**B5.** *Let $R_1, R_2, R_3, a, b, c, d \in L$. Construct the finitely presented semigroup satisfying the following condition: every word not equal to null has the form $a^{n_1} R_1 b^{n_2} R_2 c^{n_3} R_3 d^{n_4}$, where $n_i \geq 0$.*

**B6.** *Let $a, b \in L$. Construct the finitely presented semigroup satisfying the following condition: if the number of the $a$ occurences of to the word $W \neq 0$, $lenght(W) > 1$ is not less then the number of the $b$ occurences to the same word.*

*The function of argument $n$ equal to the number of the different words $W$, $lenght(W) < n$ is called the grouth function of the semigroup and denoted by $G_S(n)$.*

Suppose that $G_S(n)$ is a function, equivalent to some polinomial or rational function. In this case we can define the *Gelfand-Kirillov dimension.* Suppose $G_S(n) = c_1 n^\alpha + c_2 n^\beta + \ldots + c_k n^\gamma$. Let $\alpha > \beta > \ldots > \gamma$. Then $\alpha$ is called *Gelfand–Kirillov dimension.*

**B7.** *Construct the finitely presented semigroup $S$ such that Gelfand–Kirillov dimension of the $S$ is not integer.*

**B8.** *Construct the finitely presented semigroup $S$ such that Gelfand–Kirillov dimension of the $S$ is equal to* 2.5.

**B9.** *Let $\alpha > 2.5$. There is an algorithm finding all decimal digits of the $\alpha$. (This number is called constructive. For example, $\pi$ is the constructive number.) Construct the finitely presented semigroup $S$ such that Gelfand–Kirillov dimension of the $S$ is equal to $\alpha$.*