$\mathcal{A}$utomatic machines and finitely presented groups.

The correction.

B6. *Let $a, b, C, D \in L$. Let $Occ_a(W)$ be the number of a occurencies to the word W. Construct the finitely presented semigroup satisfying the following properties:*

i) if the word W has the form $C...D$ then $Occ_a(W) \geq {Occ_b}^2(W)$

ii) for any natural number N there exists the word $W = C...D$ such that $Occ_a(W) > N$ and $Occ_b(W) > N$.

Additional problems.

Let $n = 2^{k_1} * 3^{k_2} * \ldots * {p_s}^{k_s}$. Then n has presented the set $(k_1, k_2, \ldots, k_n)$. Now we are going to discuss an approuch for the "tree-problems". We should study how to use the massives.

A8. *Make these transitions:*

i) $2^n * 3^m --> 2^n * 3^m * 5^{k_m}$, *where k_m is the power of m-th prime in the n decompositon to the prime factors;*

j) $2^n --> 2^n * 3^s$, *где s – the number of primes in the n decompositon to the prime factors;*

k) Prove that the transition from n to n^2 is impossible.*

Hint. Consider the two cases: 1) the robot comes to the corner close enouth; 2) the robot does not come to the corner.

A13.5. *Let $\epsilon > 0$. Prove that we can instruct the soldiers (problem A13) such that they will fire in time less then $(2 + \epsilon)N$.*

Let us state some additional problems about the robot on a cellular plane.

There are some latin letters in the cells. The robot can read them but he cannot change them.

A15. *a) Can the robot without any pebble investigate the plane if we have written some letters in the cells? (We can choose how to write the letters.)*

Can the robot investigate the tape if we have written some letters in the cells? (We can choose how to write the letters.)

b) the robot using no pebbles;

c) the robot using one pebble.

Now the robot can brush cells to the black. He cannot brush them to the white. The robot moves on the square $2n + 1 * 2n + 1$. The robot can look the borders then he is on the edge. In the begining the robot is in the corner.

A16. *a) Can the robot brush all the cells except the central cell?*

b) Can the robot brush the central cell only?

The semigroup S is called *group* if the following conditions hold:

1) there exist an *unite element* e, such that for all $a \in S$ we have $ae = ea = a$;

2) for all a there exists *an inverse element* a^{-1} such that $aa^{-1} = a^{-1}a = e$.

It is easy to see that there exists unique unite. The group is said to be *commutative* if for any two elements a, b the following proposition holds: $ab = ba$.

The commutative groups are also known as *abelian* groups.

A17. *Construct the nonabelian group.*

The group is called *cyclic* if there exists a such that for any b there exists n such that $a^n = e$.

Example. Consider the residues mod. n. These residues are form a cyclic group under addition. The zero residue is unite element.

The number of group's elements is called *order* of the group and is denoted by $|G|$.

A18. *Suppose that the order of a group is prime. Prove that the group is cyclic.*

The element a of the group is called *periodic* if there exists n (*order of the element*) such that $a^n = e$. Элемент группы a называется *периодическим*, если существует некоторое натуральное n – *порядок элемента*, такое что $a^n = e$. A group is called *periodic*, if all the elements are periodic.

Burnside problem. *Does there exist a finetely generated periodic infinite group?*

The examples of these groups are well known. We should not prove it. We will use this result below.

Let us have a finite alphabet $\{g_1, g_1^{-1}, g_2, g_2^{-1} ... g_s, g_s^{-1}\}$.

It is easy to see that the words are form a semigroup under word attaching. This group is called *free group defined by n generators.*

Let us construct *a graph of the group.* A vertices of the graph are the elements of the group. The edges are said to be connect the elements a, b, if the following condition holds: $a = b * g_i$.

A19. *Draw a graph for the free abelian group with two generators.*

A20. *Draw a graph for the free nonabelian group with two generators.*

Suppose the robot moves by the graph of the infinite periodic group.

A21. *Prove that the robot cannot go away from some finite area.*

a) the robot has no pebble;

b) the robot has 1 *pebble;*

c) the robot has 2 *pebbles;*

d) the robot has n *pebbles.*

A22. *Suppose that there exist a periodic element in the group. Prove that the robot with* 3 *pebbles can investigate the graph of the group.*

The following problem has been added to make clear the constructions below.

B6.5. *Let* $R, C, D, P, Q, a, b, t_1, t_2 \in L$. *Construct a finitely presented semigroup such that the following propositions hold:*

1) if a word is not equal to null and containg C, R, D *then it is equal to* $a^{n_1}Ca^{n_2}Pa^{n_3}t_iRb^{n_4}Qb^{n_5}Db^{n_6}$, *for some* $n_i \geq 0$.

2) if a word is not equal to null and containg P, Q, D *has a form* W_1PRQW_2.

3) if a word is not equal to null and has a form CW_1RW_2D *then* W_1 *does not contain* b, W_2 *does not contain* a, W_1 *and* W_2 *contain the equal numbers of* a *and* b.

Section C. The nilpotent semigroups.

Now we shall give the following definitions. The semigroup with null is called *nilsemigroup* if for any element x there exists natural n, such that $a^n = 0$. The semigroup with null is called *nilpotent semigroup* if there exists natural n, such that for any elements $x_1, x_2, \ldots x_n$ the following condition holds: $x_1 * x_2 * \ldots * x_n = 0$

The nilpotence is an important topic in algebra. We will discuss this topic for finitely presented semigroups. Four problems are independent ones. C1 is a well-known olimpiad question. C2 is the same question in semigroup language. C3 and C4 are dealing with some intresting constructions. The other problems define an approuch to the primary project problem.

C1. *In Robotland some words are called censored. There are finite number of censored words. If some word contains a censored subword then the whole word are censored. There exist noncensored words of any lenght. Prove that there exist an infinite periodic noncensored word.*

C2. *Consider a finitly presented nilsemigroup. Suppose that all defining relations are* $W_i = 0$, *where* W_i *are some words. Prove that the semigroup is finite.*

C3. *Let $L = \{a, b\}$ be an alphabet. Let N be a natural number. Prove that there exist a word W satisfying the following conditions:*

i) lenght$(W) > N$;

ii) there are no subword of W has the form AAA.

C4. *Let $L = \{a, b, c\}$ be an alphabet. Let N be a arbitrary natural number. Prove that there exist a word W satisfying the following conditions:*

i) lenght$(W) > N$;

ii) there are no subword of W has the form AA.

Addition. Compare C3 and C4 with B3.

Hence we can construct some infinite nilsemigroup. (State the defining relations $AA = 0$ for all A). This semigroup is not finitely presented.

Let $a, b, c, d \in L$. Let us denote $x = a$, $[x] = b$, $[x = c$, $x] = d$. So, x, $[x]$, $[x$, $x]$ are the letters in the alphabet.

Let us construct the set of words L_i.

$L_1 = [x]$, $L_2 = [xx]$, $L_3 = [xx[x]x]$,

$$L_n = [xxL_1xL_2L_1xL_3L_2L_1x \dots L_{n-2}L_{n-3} \dots L_2L_1x]$$

.

C5. *Prove that L_i contain no square words (AA).*

C6. *Let $x, [x], [x, x] \in L$ and $o, [o], [o, o] \in K$. Construct the infinite sets $A_1, A_2, \dots$ on L and $B_1, B_2, \dots$ on K and a finite system of defining relations such that the following conditions hold: $A_iB_j = B_jA_i$, where $|i - j| > 1$, but $A_iB_j \neq B_jA_i$, where $|i - j| \leq 1$.*

C7. *Let $x, [x], [x, x] \in L$ and $o, [o], [o, o] \in K$. Construct the infinite sets $A_1, A_2, \dots$ on L and $B_1, B_2, \dots$ on K and a finite system of defining relations such that the following conditions hold: $A_iB_j = B_jA_{i+j}$, where $|i - j| > 1$.*

C8. *Let $x, [x], [x, x] \in L$ Construct the infinite sets $A_1, A_2, \dots$ on L and a finite system of defining relations such that the following conditions hold: $A_iA_j = A_jA_i$, where $|i - j| > 1$, but $A_iA_j \neq A_jA_i$, where $|i - j| \leq 1$*

C9. *Let $x, [x], [x, x] \in L$ Construct the infinite sets $A_1, A_2, \dots$ on L and a finite system of] defining relations such that the following conditions hold: $A_iA_j = A_jA_i$, where $|i - j| > 1$ and $A_iA_{i-1}A_i = 0$.*

C10. *Construct a finitely presented infinite nilsemigroup.*