Domino tilings

С. Берлов, К. Кохась

2 августа 2004 г.

Problems

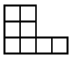
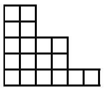
1 *Parity*

1.1. Prove that the number of ways to tile the $n \times n$ square is even.

1.2. Prove that for $n > 2$ the number of ways to tile the $n \times n$ square with dominoes is divisible by 4.

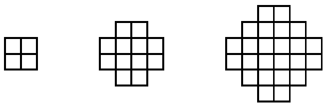
1.3. Prove that the number of ways to tile $3 \times 2n$ and $7 \times 2n$ rectangles is odd.

1.4. Consider the following figures

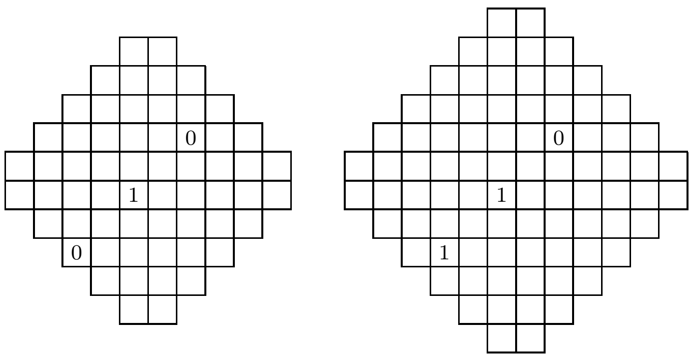
$H_1 =$  , $H_2 =$  , $H_3 =$  , ...

H_n is defined from H_{n-1} by adding a grid of size $2 \times (2n-1)$ to the left of H_{n-1}. Prove that the number of domino tilings of H_n is odd.

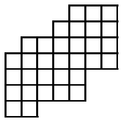
Aztec diamond of order n is a "rombus" of side n on the square lattice. See examples for $n = 1, 2, 3$.



1.5. Consider a "midline" of Aztec diamond, i.e. a line of slope 1 running through the center of the Aztec diamond. It passes through n squares. Number these squares form lower left to upper right and place the numbers 0 and 1 in the odd squares arbitrarily. See in the picture the example for diamonds of odd and even order. Consider domino tilings of Aztec diamond that correspond to this marking in the following sense. For any domino that cover a square with a number, its second square is upper or to the left of the first one if the number equals 0, and is below or to the right if the number equals 1. Prove that for all placements of numbers on the midline the numbers of domino tilings that corresponds to the placement are the same. Particularly, the number of all domino tilings of the Aztec diamond of order n is divisible by $2^{\lfloor n/2 \rfloor}$.



1.6. a) Prove that the following figure has even number of domino tilings.



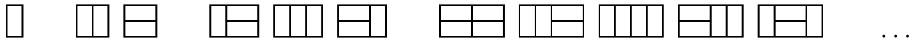
b) Prove that the number of tilings of the Aztec diamond of order n is divisible by 2^n.

1.7. Prove that the $n \times (n+1)$ rectangle has odd number of domino tilings.

2 Recurrence relations

Let $N(n,m)$ denotes a number of domino tilings of $n \times m$ rectangle and f_n denotes Fibonacci numbers: $f_0 = f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$.

2.1. Prove that $N(n,2)$ is the Fibonacci sequence.



1 2 3 5

2.2. Prove combinatorially that $f_{n+m} = f_n f_{m-1} + f_n f_m$.

2.3. Prove that the sequence $N(n,3)$ satisfies the recurrence relation

$$N(n,3) = 4N(n-2,3) - N(n-4,3)\,.$$

2.4. Two squares A and B are marked on the boundary of 1993×1993 rectangle. Prove that the number of domino tilings of the rectangle without square A equals to the number of domino tilings of the rectangle without square B.

2.5. Let d be the number of domino tiling of the 100×200 rectangle; d_2 be the number of domino tiling the same rectangle without two corner squares adjacent to the same side; d_4 be the number of domino tiling the rectangle with four corner squares deleted. Prove that

$$d \cdot d_4 = 2d_2^2\,.$$

2.6. Consider a triangle of numbers

$$\begin{array}{ccccccccc} a & & c & & e & & g & & \ldots \\ & b & & d & & f & & \ldots & \\ & & h & & k & & \ldots & & \\ & & & \ell & & \ldots & & & \\ & & & & \ldots & & & & \end{array} \tag{1}$$

which is constructed by the rule: for any four numbers that form a “rombus” the following relation holds.

$$\begin{array}{ccc} & w & \\ x & & y \\ & z & \end{array} \qquad wz = xy + 1\,. \tag{2}$$

Let the first row of the triangle filled with numbers 1, and the second row filled with 2’s. Find all other numbers in the triangle.

White magic. Consider a triangle from the previous problem. Assume that the entries of the first and second rows are independent indeterminates, and all other entries are rational functions on these inderminates constructed by the formula (2). For example, it is easy to calculate that the element ℓ in the triangle (1) has a form

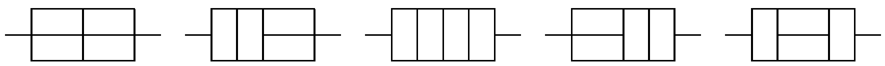
$$\ell = \frac{hk+1}{d} = \frac{\frac{bd+1}{c}\cdot\frac{df+1}{e}+1}{d} = bc^{-1}de^{-1}f + c^{-1}e^{-1}f + c^{-1}d^{-1}e^{-1} + bc^{-1}e^{-1} + d^{-1}\,. \tag{3}$$

Now consider all domino tilings of 2×4 rectangle and fulfill the following steps.

1) Add a horizontal segment to both short sides of the rectangle;

2) For each lattice vertex in the midline of the rectangle write down its reduced valency, i.e. the number of unit segments of tiling adjacent to this vertex minus 3.

3) take this ordered set of 5 numbers as a set of exponents in the product $b^*c^*d^*e^*f^*$.



...

1 −1 1 −1 1	0 −1 0 −1 1	0 −1 −1 −1 0	1 −1 0 −1 0	0 0 −1 0 0
$bc^{-1}de^{-1}f$	$c^{-1}e^{-1}f$	$c^{-1}d^{-1}e^{-1}$	$bc^{-1}e^{-1}$	d^{-1}

Compare the result with r.h.s. of (3).

2.7. Prove that “White magic” is a general fact, i.e.

1) rational functions in triangle (1) are sums of monomials, that are products of the initial indeterminates in powers ± 1 or 0. All the coefficients in the sum equal 1.

2) The monomials of the function in the $(n+2)$-th row are in one-to-one correspondence with the domino tilings of $2\times 2n$ rectangle by the rule above.

3 Chow-chow

3.1. Is it true that for any n there exists a figure which has exactly n domino tilings?

3.2. Prove that if a polygon without holes can be tiled by dominoes then it has at least 1 side of even length.

3.3. Prove that $(5/4)^{mn} < N(m,n) < (10/7)^{mn}$.

3.4. Prove that the number of domino tilings of $3\times 2n$ rectangle has a form a^2+2b^2.

3.5. Consider a domino tiling of the $2n\times 2m$ rectangle. Let a checker move on this rectangle from left to right by the following rule. It starts at the left side of the rectangle. When it enters (from the left) some square of next domino it moves to the second square of the domino and goes out (to the right). Prove that a checker will finish its movement on the same horizontal line as it starts.