4 *After Semifinal*

2.5. Prove combinatorially that Fibonacci numbers satisfy the relation

$$f_{2n+2}f_{2n-2} = f_{2n}^2 + 1\,.$$

3.3. Prove that there exist constants C_1, C_2, such that for all even m and n an inequality

$$C_1(5/4)^{mn} < N(m,n) < C_2(10/7)^{mn}$$

holds.

An $n \times n$ square with some cells of its upper row and right column being deleted is called an $n-$block. Let us enumerate the rows of n–block from bottom to top and its columns from left to right by numbers 1, 2, $\ldots n$. Let us denote a cell by two coordinates, i.e. the number of its row and the number of its column. We call an n–block *good* if for any two cells (n,i) and (i,n), $1 \leqslant i \leqslant n-1$, exactly one cell is deleted and the cell (n,n) is also deleted.

4.1. Prove that an $n-$block has an odd number of domino tilings if and only if it is good.

4.2. Prove that the number of domino tilings of an $n \times (2n+1)$ rectangle is divisible by 4.

4.3. Prove that the number of domino tilings of a $(2n+1) \times (2n+1)$ square $(n > 1)$ with central cell deleted is divisible by 4.

4.4. Prove that the number of domino tilings of an $m \times n$ rectangle is odd if and only if numbers $m+1$ and $n+1$ are coprime.

4.5. Prove that the number of domino tilings of an $n \times 2n$ rectangle has residue 1 modulo 4.

4.6. Prove that the number of domino tilings of a $2n \times 2n$ square $(n > 2)$ is divisible by 8.

4.7. Prove that the number of domino tilings of a $2n \times 2n$ square $(n > 3)$ is divisible by 16.

4.8. Consider a figure that is symmetrical with respect to a line with slope 1 that passes through the centers of grid cells. Suppose that this line contains the centers of exactly n cells of the figure. Prove that the number of domino tilings of this figure is divisible by $2^{n/2}$.

4.9. Prove that the number of domino tilings of a figure is even if and only if it is possible to mark a nonempty set of its cells such that each cell (either marked or unmarked one) has an even number of marked neighbours.

4.10. Prove that the number of domino tilings of an $2n \times 2n$ square has a form $2^n(2k+1)^2$.

4.11. Let $2k+1 = g.c.d.(2m+1, 2n+1)$. Prove that the number of domino tilings of a $2n \times 2m$ rectangle is divisible by 2^k but not by 2^{k+1}.

4.12. Let T_n denote the number of domino tilings of Aztec diamond of order n. Prove that

$$T_{n+1}T_{n-1} = 2T_n^2\,.$$

By the way, it follows that $T_n = 2^{n(n+1)/2}$.

4.13. The cell $a1$ of an $(n+1) \times (n+1)$ chessboard is deleted. A semiking is a chess figure that can do moves to one cell right, down or right-down. A semiking stands on each cell of the line a of the chessboard. How many sets of n nonintersecting semiking trajectories that join the cells of the line "a" with the cells of the bottom line of the chessboard do exist?