Domino tilings

S.Berlov, K.Kokhas

Jury team: S.Berlov, I.Bogdanov, K.Kokhas, K.Kuyumzhiyan

Problems

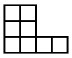
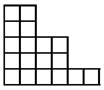
1 Parity

1.1. Prove that the number of ways to tile an $n \times n$ square is even.

1.2. Prove that for $n > 2$ the number of ways to tile an $n \times n$ square with dominoes is divisible by 4.

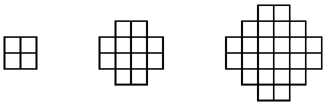
1.3. Prove that the numbers of ways to tile $3 \times 2n$ and $7 \times 2n$ rectangles are odd.

1.4. Consider the following figures:

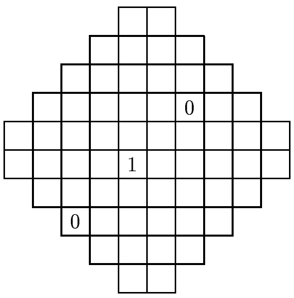
$H_1 =$  , $H_2 =$  , $H_3 =$  , ...

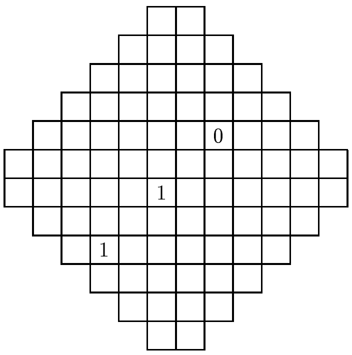
(H_n is obtained by adding a grid of size $2 \times (2n-1)$ to the left of H_{n-1}). Prove that the number of domino tilings of H_n is odd.

Aztec diamond of order n is a "rombus" of side n on the square lattice. See examples for $n = 1, 2, 3$.

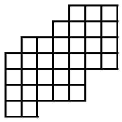


1.5. Consider a "midline" of Aztec diamond, i.e. a line of slope 1 running through the center of the Aztec diamond. It passes through $2\lceil n/2 \rceil$ squares. Number these squares from lower left to upper right and place the numbers 0 and 1 in the odd squares arbitrarily. See examples for diamonds of odd and even orders in the picture. Consider domino tilings of Aztec diamond that correspond to this marking in the following sense. For any domino that covers a square with a number, its second square is upper or to the left of the first one if the number equals 0, and is below or to the right if the number equals 1. Prove that for any placement of numbers on the midline the number of domino tilings that correspond to this placement is the same. Particularly, the number of all domino tilings of the Aztec diamond of order n is divisible by $2^{\lceil n/2 \rceil}$.





1.6. a) Prove that the following figure has even number of domino tilings:



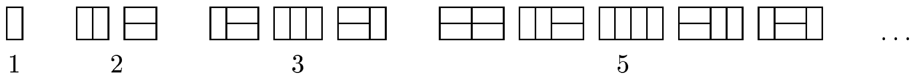
b) Prove that the number of tilings of the Aztec diamond of order n is divisible by 2^n.

1.7. Prove that an $n \times (n+1)$ rectangle has odd number of domino tilings.

2 *Recurrence relations*

Let $N(n,m)$ denote a number of domino tilings of an $n \times m$ rectangle and f_n denote an nth Fibonacci number: $f_0 = f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$.

2.1. Prove that $N(n,2)$ is the Fibonacci sequence:



2.2. Prove combinatorially that $f_{n+m} = f_{n-1}f_{m-1} + f_nf_m$.

2.3. Prove that the sequence $N(n,3)$ satisfies the recurrence relation

$$N(n,3) = 4N(n-2,3) - N(n-4,3).$$

2.4. Two cells A and B are marked on the boundary of 1993×1993 rectangle so that the number of cells between them is odd. Prove that the number of domino tilings of the rectangle without square A equals to the number of domino tilings of the rectangle without square B.

2.5. Let d be the number of domino tilings of the 100×200 rectangle; d_2 be the number of domino tilings of the same rectangle without two corner squares adjacent to the same side; $\widetilde{d}_2$ be the number of domino tilings of the rectangle without two corner squares adjacent to another side; d_4 be the number of domino tilings of the rectangle with four corner squares deleted. Prove that

$$d \cdot d_4 = d_2^2 + \widetilde{d}_2^2.$$

2.6. Consider a triangle of numbers

$$\begin{array}{ccccccccc} a & & c & & e & & g & & \dots \\ & b & & d & & f & & \dots & \\ & & h & & k & & \dots & & \\ & & & \ell & & \dots & & & \\ & & & & \dots & & & & \end{array} \tag{1}$$

which is constructed by the following rule. For any four numbers that form a “rombus” the following relation holds:

$$\begin{array}{ccc} & w & \\ x & & y \\ & z & \end{array} \qquad wz = xy + 1. \tag{2}$$

Let the first row of the triangle be filled with numbers 1, and the second row be filled with 2’s. Find all other numbers in the triangle.

White magic. Consider a triangle from the previous problem. Assume that the entries of the first and second rows are independent indeterminates, and all other entries are rational functions on these indeterminates constructed by the formula (2). For example, it is easy to calculate that the element ℓ in the triangle (1) has a form

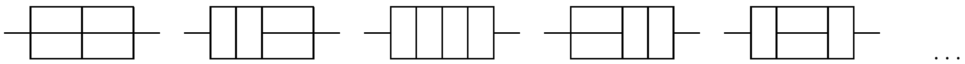
$$\ell = \frac{hk+1}{d} = \frac{\frac{bd+1}{c}\cdot\frac{df+1}{e}+1}{d} = bc^{-1}de^{-1}f + c^{-1}e^{-1}f + c^{-1}d^{-1}e^{-1} + bc^{-1}e^{-1} + d^{-1}\,. \quad (3)$$

Now consider all domino tilings of a 2×4 rectangle and perform the following steps.

1) Add a horizontal segment to both short sides of the rectangle;

2) for each lattice vertex in the midline of the rectangle write down its reduced valency, i.e. the number of unit segments of tiling adjacent to this vertex minus 3.

3) take this ordered set of 5 numbers as a set of exponents in the product $b^*c^*d^*e^*f^*$.



$1\ {-1}\ 1\ {-1}\ 1$	$0\ {-1}\ 0\ {-1}\ 1$	$0\ {-1}\ {-1}\ {-1}\ 0$	$1\ {-1}\ 0\ {-1}\ 0$	$0\ 0\ {-1}\ 0\ 0$
$bc^{-1}de^{-1}f$	$c^{-1}e^{-1}f$	$c^{-1}d^{-1}e^{-1}$	$bc^{-1}e^{-1}$	d^{-1}

Compare the result with r.h.s. of (3).

2.7. Prove that "White magic" is a general fact, i.e.

1) rational functions in triangle (1) are the sums of monomials that are the products of the initial indeterminates with powers ± 1 or 0. All the coefficients in the sum equal 1.

2) The monomials of the function in the $(n+2)$-th row are in one-to-one correspondence with the domino tilings of $2\times 2n$ rectangle by the rule above.

3 *Chow-chow*

3.1. Is it true that for any n there exists a figure which has exactly n domino tilings?

3.2. Prove that if a polygon without holes can be tiled by dominoes then it has at least one side of even length.

3.3. Prove that there exist constants C_1, $C_2 > 0$ such that for all even m and n an inequality

$$C_1(5/4)^{mn} < N(m,n) < C_2(10/7)^{mn}$$

holds.

3.4. Prove that the number of domino tilings of a $3\times 2n$ rectangle has a form a^2+2b^2.

3.5. Consider a domino tiling of the $2n\times 2m$ rectangle. Let a checker move on this rectangle from left to right by the following rule. It starts at the left side of the rectangle. When it enters (from the left) some square of next domino it moves to the second square of the domino and goes out (to the right). Prove that a checker will finish its movement on the same horizontal line as it starts.

4 *After Semifinal*

2.6(new). Prove combinatorially that Fibonacci numbers satisfy the relation

$$f_{2n+2}f_{2n-2} = f_{2n}^2 + 1.$$

An $n \times n$ square with some cells of its upper row and right column deleted is called an n-block. Let us enumerate the rows of n-block from bottom to top and its columns from left to right by numbers 1, 2, $\dots n$. Let us denote a cell by two coordinates, i.e. the number of its row and the number of its column. We call an n-block *good* if for any two cells (n, i) and (i, n), $1 \leqslant i \leqslant n-1$, exactly one cell is deleted and the cell (n, n) is also deleted.

4.1. Prove that an n-block has an odd number of domino tilings if and only if it is good.

4.2. Prove that the number of domino tilings of an $n \times (2n+1)$ rectangle is divisible by 4.

4.3. Prove that the number of domino tilings of a $(2n+1) \times (2n+1)$ square $(n > 1)$ with central cell deleted is divisible by 4.

4.4. Prove that the number of domino tilings of an $m \times n$ rectangle is odd if and only if numbers $m+1$ and $n+1$ are coprime.

4.5. Prove that the number of domino tilings of an $n \times 2n$ rectangle has residue 1 modulo 4.

4.6. Prove that the number of domino tilings of a $2n \times 2n$ square $(n > 2)$ is divisible by 8.

4.7. Prove that the number of domino tilings of a $2n \times 2n$ square $(n > 3)$ is divisible by 16.

4.8. Consider a figure that is symmetrical with respect to a line with slope 1 that passes through the centers of grid cells. Suppose that this line contains the centers of exactly n cells of the figure. Prove that the number of domino tilings of this figure is divisible by $2^{n/2}$.

4.9. Prove that the number of domino tilings of a figure is even if and only if it is possible to mark a nonempty set of its cells such that each cell (either marked or unmarked one) has an even number of marked neighbours.

4.10. Prove that the number of domino tilings of a $2n \times 2n$ square has a form $2^n(2k+1)^2$.

4.11. Let $2k+1 = g.c.d.(2m+1, 2n+1)$. Prove that the number of domino tilings of a $2n \times 2m$ rectangle is divisible by 2^k but not by 2^{k+1}.

4.12. Let T_n denote the number of domino tilings of Aztec diamond of order n. Prove that

$$T_{n+1}T_{n-1} = 2T_n^2\,.$$

By the way, it follows that $T_n = 2^{n(n+1)/2}$.

4.13. The cell $a1$ of an $(n+1) \times (n+1)$ chessboard is deleted. A semiking is a chess figure that can do moves to one cell right, down or right-down. A semiking stands on each cell of the line a of the chessboard. How many sets of n nonintersecting semiking trajectories that join the cells of the line "a" with the cells of the bottom line of the chessboard do exist?

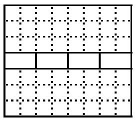
Solutions

1.1. The set of all tilings consists of pairs of tilings that are symmetrical with respect to the main diagonal.

1.2. Consider a diagonal that goes from the lower left corner to the upper right one. We will prove that there exists a 2×2 square that consists of two dominoes and contains two cells of the diagonal. Suppose that there is no such square. Then the domino which contains the lower left corner cell contains also (of course) its upper or left neighbours. The next domino contains also upper or left neighbours of the second cell of the diagonal. And so on. When we reach the upper right corner we obtain a contradiction.

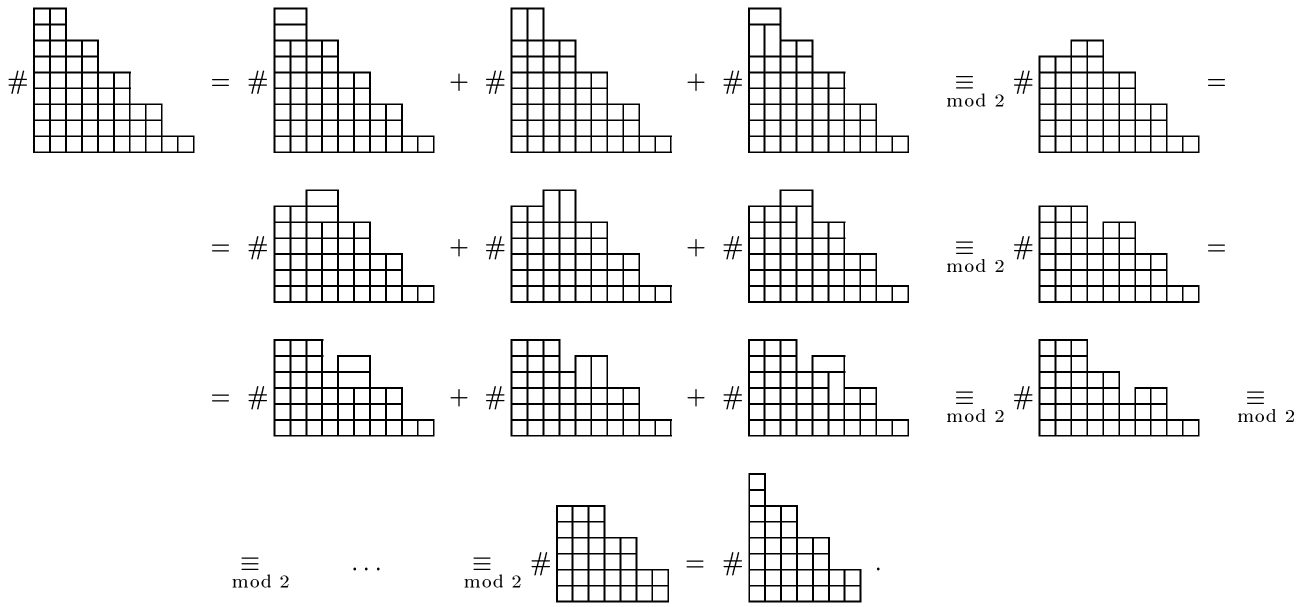
Mark the lowest diagonal 2×2 square in each tiling. Then for any position of the diagonal 2×2 square there exist an even number of tilings that contain a marked diagonal 2×2 square in this position (this is due to the symmetry with respect to the diagonal). But we can also choose one of two ways to tile this square. So the total number of tilings is divisible by 4.

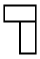
1.3. The statement about a $3 \times 2n$ rectangle can be easily proved by induction (see formula (4) in the solution 2.3). To prove the second statement mark the midpoints of sides of length 7 and consider a symmetry with respect to the corresponding midline. Then each tiling has a pair (i.e. the tiling that is symmetrical with respect to the midline) except those tilings for which this midline is covered by n full dominoes and the rectangles $3 \times 2n$ are tiled symmetrically to each other:



The number of the latter tilings equals the number of tilings of a $3 \times 2n$ rectangle. So it is odd, and the total number of tilings is odd, too.

1.4. We have found this statement and its proof in [9]. Induction by n. The base of induction ($n = 1, 2$) is trivial. Let us investigate how are the dominoes situated along the upper boundary of the figure H_n.

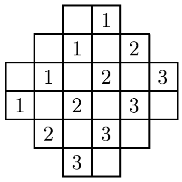


In the first line of the formula we have removed a 3×2 rectangle part of the figure, while in each next line we remove a part of the form . At the end of the formula we obtain a figure symmetrical to H_{n-1}.

1.5. The statement is taken from [12]. In fact, the proof is contained in [9]. The main tool is the construction of problem 4.8 and the proof is analogous.

1.6. a) Analogously to problem 1.2, the figure contains a diagonal line of cells from the lower left corner to the upper right one. So we can split the whole sets of tilings onto pairs by flipping the dominoes in the lowest diagonal 2×2 square.

б) Aztec diamond of order n contains n independent diagonal lines (see example for $n = 3$).



Each line determines its 2×2 square and these squares do not intersect. For each tiling consider 2^n tilings (with initial tiling included) that differ by flips of dominoes in these 2×2 squares only. This fact gives us a partition of the whole set of tilings into subsets of 2^n elements.

1.7. We found this problem of Dmitry Karpov in [1, problem 97.109].

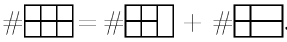
Consider a diagonal line of cells from the upper left corner of the rectangle to the cell that is neighbour of the lower right corner. It is sufficient to consider only those tilings which do not contain a diagonal 2×2 square. We will show that there exists a one-to-one correspondence between the set of all tilings of $n \times (n+1)$ rectangle and the set of such tiling of $(n-1) \times n$ rectangle.

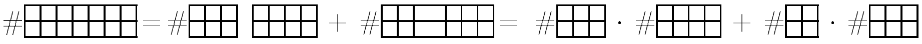
To define this correspondence we remove from the $n \times (n+1)$ rectangle all the dominoes that contain cells of the diagonal constructed before. After that we move the remaining upper part of the figure down and left by one cell. We obtain an $(n-1) \times n$ rectangle (prove this!).

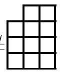
To describe the inverse mapping consider an analogous diagonal line of cells in $(n-1) \times n$ rectangle. Consider separately all the dominoes that are "not upper" than this diagonal line and move all other dominoes up and left by one cell. Then there is a unique way to tile a gap between these two parts to obtain a tiling of $n \times (n+1)$ rectangle.

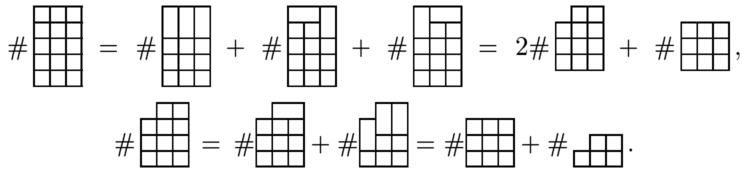
Now the statement can be easily proved by induction.

We can obtain another solution using the solution of problem 4.1. For this we can remark that $n \times (n+1)$ rectangle is a good $(n+1)$-block.

2.1. Note that  Hence the relation $N(n,2) = N(n-1,2) + N(n-2,2)$ holds.

2.2. Analogously. See [4].  Hence $f_{n+m} = f_n f_{m-1} + f_n f_m$.

2.3. [4] Let $N_n = N(n,3)$ and let $\gamma_n =$ #  ($n-2$ full rows and one not full). Note that



We can summarize our obsevations as follows

$$N_n = 2\gamma_n + N_{n-2}\,,$$
$$2\gamma_{n-2} = N_{n-2} - N_{n-4}\,,$$
$$2\gamma_n = 2\gamma_{n-2} + 2N_{n-2}\,.$$

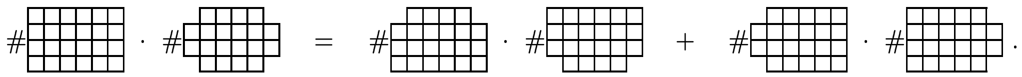
We sum up these three equalities and obtain the statement of our problem.

Just in case, we can solve the recurrence relation and obtain the formula ([4])

$$N(n,3) = \begin{cases} 0 & \text{for odd } n\,, \\ \frac{\sqrt{3}}{6}\left((\sqrt{3}+1)(2+\sqrt{3}\,)^{n/2} + (\sqrt{3}-1)(2-\sqrt{3}\,)^{n/2}\right) & \text{for even } n\,. \end{cases}$$

2.4. This problem is found in [3]. Consider a chessboard colouring. Let the corners be black. Then we can assume that both cells A and B are black (if they both are white then we have 0 tilings in both cases). Consider an arbitrary tiling of the whole rectangle T without cell A and choose the domino that contains the cell $B = B_1$. Let B_2 be the second cell of this domino. Move this domino so that the cell B_1 moves onto B_2. Let B_3 be the new cell covered by this shifted domino. Consider the domino in our tiling that contains the cell B_3. Let B_4 be the second cell of this domino. Move the domino in the direction from B_3 to B_4 and so on. We shall get a chain of dominoes, which may end either if it cycle or if it leads to B_1 cell (check that one can always move a domino in this direction; this follows from the fact that the coordinates of all B_{2k+1} cells have the same parity as the corresponding coordinates of B_1). Note that cycling is also impossible since the cycle interior consists of odd number of cells (the proof is left to the reader). Hence, our chain ends in A. Then we can reconstruct our tiling for $T \setminus A$ into a tiling of $T \setminus B$ by recombining this chain so that A is covered and B is not. Hence, for each tiling of $T \setminus A$ we find a corresponding tiling of $T \setminus B$. Since there is an analogous inverse transformation, we are done.

2.5. This statement is a weaker version of Theorem 2.1 in [7]. We will prove the following oriented version:



Consider a tiling of the whole rectangle and a tiling of the rectangle without corner cells. As in the previous solution, we take one of the corner cells and construct a chain of dominoes which may end only in another corner cell (these cells must be adjacent to one side; use the parity reason). An analogous chain beginning in the opposite corner cell must end in the remaining corner, and these chains will not intersect. Let us recombine the first of these chains. We obtain two tilings of one of the types showed in the right part of the formula. The inverse transformation is defined analogously. Hence the transformation constructed is a bijection.

Note. This solution remains valid in a more general situation. Let D be a connected cell figure, and a, b, c, d be its border cells (a and c are black, b and d are white in chess colouring; a, b, c, d occur in exactly this order on the border). Then

$$N(D)N(D \setminus \{a,b,c,d\}) = N(D \setminus \{a,b\})N(D \setminus \{c,d\}) + N(D \setminus \{a,d\})N(D \setminus \{b,c\})\,.$$

2.6. Answer: these are Fibonacci numbers f_{2n}: 1 2 1 5 2 1 13 5 2 1 … It follows from the equality

$$f_{2n+2}f_{2n-2} = f_{2n}^2 + 1\,.$$

This equality follows from the remark in the solution 2.5 . To see this consider the marked cells on the following picture.

	a					d	
	b					c	

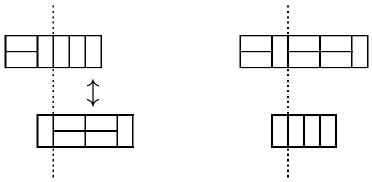
We will give an independent combinatorial proof of this formula.

Consider all possible (ordered) pairs of tilings of a $2 \times 2n$ rectangle, and also pairs consisting of a tiling of a $2 \times (2n+2)$ rectangle and a tiling of a $2 \times (2n-2)$ rectangle. See the example of such pairs and a convenient way to arrange them with respect to each other on the picture.

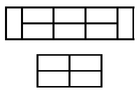
Define a one-to-one correspondence between these sets of pairs. During the construction we will have to add one more element to the first set.

The correspondence looks as follows. If the lower rectangle has a cut that cuts off a 2×2 square from the right (i.e. this square is tiled with two whole dominoes), then we move this square upwards and glue it to the upper rectangle. This case is drawn above.

In the case when this square is not cut, i.e. the line containing right side of the upper rectangle passes through some domino in the lower rectangle, we will try to find a suitable cut to the left from this line. We will check all vertical lines from right to left until we find one of them that cuts off two rectangle parts tiled by whole dominoes both in upper and lower rectangles. It is easy to see that there always exists such a cut (the leftmost position for the cut is the left side of the lower rectangle). When the cut is found we change the right parts of the rectangles that are cut off by the cut.



It is clear that this correspondence maps different pairs of tilings of a $2 \times 2n$ rectangle into different pairs of tilings of $2 \times (2n+2)$ and $2 \times (2n-2)$ rectangles. This is because we can easily describe an inverse mapping: look at the lower $2 \times (2n-2)$ rectangle from right to left trying to find a line that cuts off no domino in both rectangles, and after that we change right parts of rectangles that is cut off by this line. The joint cut does not exist in the unique case that is shown on the picture.



2.7. We found this statement in the paper of J.Propp [11], where he does not prove it. His proof seems to be nonelementary. But since we know combinatorial interpretation of function in the triangle, we can give a combinatorial proof.

To prove the statement it is sufficient to show that rational functions constructed by the algorithm "White magic" satisfy "rombus recurrence":

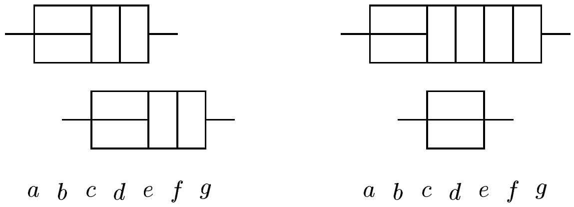
$$\begin{array}{ccc} & w & \\ x & & y \\ & z & \end{array} \qquad wz = xy + 1\,. \tag{5}$$

To do this we will use the one-to one correspondence constructed in the previous problem. We will consider particular case $n = 4$, that is general enough to see all the details.

So, consider a functional triangle

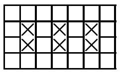
$$\begin{array}{ccccccccccc}
\lambda & & b & & d & & f & & \ldots & & \\
 & a & & c & & e & & g & & \ldots & \\
 & & h & & k & & \ell & & \ldots & & \\
 & & & p & & q & & \ldots & & & \\
 & & & & r & & \ldots & & & &
\end{array}$$

which consists of rational functions in variables a, b, c, d, e, f, g, $\ldots$, obtained by the algorithm "White magic". Prove that the equation $kr = pq + 1$ holds. Each function in this equation is a sum of terms that correspond to domino tilings of some rectangles. In order to calculate the product kr we have to take all pairs of 2×4 rectangle domino tilings, multiply terms corresponding to each pair, and after that sum up all the results. We can calculate the product pq analogously. Remark that we have to takes variables in terms with respect to positions of functions that we multiply in the triangle. See example of pairs of tilings and lists of variables that correspond to vertical lines.



Now observe that the one-to-one correspondence of the previous problem maps a pair of tilings into a pair that determines the same term! This is evident: the map changes parts of the picture by vertical moves. The sums of reduced valencies in each vertical line do not change during this moves, and therefore the terms coincide.

3.1. Answer: yes, it is true. For example, the following figure with n holes has $n+1$ domino tiling. ([8], $n=3$ on the example):



3.2. This solution is due to S. Dorichenko.

If the polygon has different number of white and black cells then the statement of the problem holds.

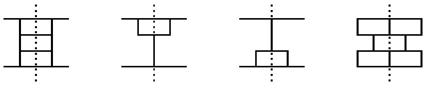
Indeed, let our polygon consist of N black and N white cells in chess colouring. Let us calculate the total perimeter of all black cells. It is equal to $4N$, and from the other hand it equals the sum of all black parts of the border plus sum of all inner grid lines. Now let us calsulate the total perimeter of all white cells. It is equal to $4N$, and from the other hand it equals the sum of all white parts of the border plus sum of all inner grid lines. Therefore the total length of black parts of the border is equal to the total length of the white part of the border. This is evidently a contradiction, because all corners of the polygon have the same colour (say, black) and then each side of the polygon has more black unit segments than white ones.

3.3. We have taken this statement from [10]. The left inequality follows from the statement of problem 2.1 and the fact that $f_n > \frac{1}{100}\left(\frac{1+\sqrt{5}}{2}\right)^n$ (one can easily check this by induction). The inequality $N(m,n) < (\sqrt{2})^{mn}$ follows from the fact that an arbitraty tiling of an arbitrary figure of area S can be encoded by the sequence consisting of $S/2$ bits by the following algorithm.

Consider the leftmost cell between all upmost cells that has not been encoded yet. We write into the encoding sequence a digit 0 or 1 depending on the orientation of the domino that contains this cell. At the end of the encoding we will obtain a sequence of $mn/2$ bits with clear rule of decoding.

It is showed in [10] that for large n $N(n,n) \approx (1.34)^{n^2}$.

3.4. Consider a midline of the rectangle that splits it into two parts $3 \times n$. We will assume that n is odd (for even n the solution is analogous). The following configurations near the midline are possible.



The last case does not correspond to a tiling because it forces the tiling of the whole strip of width 3. To cases in the middle are symmetrical to each other. Let a be a number of tilings of $3 \times (n-1)$ rectangle and b be a number of tilings of $3 \times n$ rectangle with one corner cell removed. Then the first picture has a^2 tilings and any of the second and the third ones has b^2 tilings.

3.5. Consider a chess colouring. Let $2n$ be the height and $2m$ be the width of the rectangle. Consider n checkers that move according to the rules and begin their movement on white cells outside the rectangle adjacent to its left boundary vertical line. It is clear that entering the next domino any of them visits at first a black cell and then a white cell. It is also clear that their trajectories do not intersect. Therefore the checkers will finish their movement on different white squares outside the rectangle adjacent to its right side, i.e. on the same horisontal lines as in the beginning.

It is easy to see that we can uniquely reconstruct a domino tiling if we know the trajectories of all checkers. Therefore there is a one-to-one correspondence between the set of domino tilings of $2n \times 2m$ rectangle and the set of collections of n trajectories. We have taken this observation from [13].

4.1. This statement belongs to D. Karpov [1, solution 97.109].

The cells of the rightmost column and the uppermost row of $n \times n$ square will be referred to as a frame. We prove by induction on n that the number of domino tilings of n-block is odd if and only if the block is good. The base of induction $n = 3$ is trivial. We now prove a step of induction.

Consider a good $(n + 1)$-block D. Remove all the dominoes that contain the cells of the frame. We obtain an n-block. It would be good if and only if each removed domino contains exactly one cell of the frame. Note that this way to place dominoes is unique. So there is a unique way of removing these dominoes which yields a good n-block. All the other ways yield non-good blocks. By induction hypothesis we conclude that the number of ways to tile a good $(n + 1)$-block is odd.

Now let D be a non-good $(n + 1)$-block. As above, the number of $(n + 1)$-block domino tilings is odd if and only if we can obtain an odd number of good $(n + 1)$-blocks. Consider several dominoes both cells of which belong to the frame of non-good block D. Suppose that we can obtain a good block by removing all these dominoes. By definition, good block contains exactly one of two cells symmetrical with respect to the main diagonal. Therefore a block D contains all cells which are symmetrical to the cells of all dominoes that have been removed. Then we can delete from D all the dominoes which are symmetric to ours obtaining *another* good block. So, we can obtain an even number of good blocks by removing dominoes inside a frame of non-good $(n+1)$-block. Hence the number of domino tilings of non-good $(n + 1)$-detail is even and the step of induction follows.

4.2. Consider a midline column of cells that divides the rectangle into two $n \times n$ squares. Mark all the dominoes which contain the cells of this column and remove them. We obtain two blocks that can be tiled independently. If both of them are not good then we have an even number of tilings.

Now consider a case when one of the blocks is good. Then it is an $n \times (n - 1)$ rectangle, the another block is an $n \times n$ square, its number of tilings is divisible by 4 (for $n > 2$).

Meticulous reader can check the statement for $n = 2$ by hands.

4.3. Consider a column and a row that pass through the deleted square (a "cross"). Remove all the dominoes that contain cells of this cross. We obtain 4 nonintersecting blocks that can be tiled independently. The set of tilings of the initial square obtained in this way will be referred to as a *class*.

It is sufficient to consider the cases when at least three of these four blocks are good (in all other cases divisibility by 4 is trivial). Moreover, if the deleted dominoes form a non-symmetrical figure (with respect to the center of the square), then we can split all the classes into groups containing 4 classes each (use the rotations on 90°, 180°, 270°). So in this case the total number of tilings is divisible by 4.

The only missed case is when all 4 details are good and no domino lies in the cross. The number of domino tilings of classes of this type has residue 1 (mod 16), because it is a fourth power of an odd integer. The class of this type is determined by the placements of the dominoes that has been removed from the lower vertical line of the cross (all these dominoes are horisontal and oriented to the left or to the right of the cross). Therefore the number of these classes is divisible by 2^n (and particularly by 4 if $n \geqslant 2$).

4.4. "The Euclid algorithm". Let $m > n$. Consider a grid line (an "axis") that cuts off an $n \times n$ square from our rectangle. Remove all dominoes that contain cells of this axis. Then the rest part of the rectangle is a block. It is sufficient to consider a case when this is a good block. In this case the axis intersects exactly n dominoes, one block is an $n \times (n - 1)$ rectangle and the second block is a $(m - n - 1) \times n$ rectangle for which the induction hypothesis holds.

4.5. It is easy. Consider the midline that divides the rectangle into two $n \times n$ squares and remove all the dominoes that are dissected by this line. We obtain two equal blocks. If both of them are not good, then we have a number of tilings that is divisible by 4. If they both are good then the midline dissects exactly n dominoes (so this case is unique) and the number of tilings is square of an odd integer, therefore having residue $1 \pmod 4$.

One can prove analogously that the number of tilings of $n \times (2n + 2)$ rectangle has residue $1 \pmod 4$.

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