NEW WAYS OF WEAVING BASKETS

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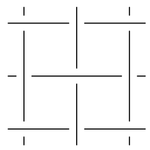
*Denn das Leben ist keine Rechnung und keine
mathematische Figur, sondern ein Wunder.
H. Hesse, Kurgast.*

Definitions

An $(m \times n)$ *weaving* is defined as an $(m \times n)$ matrix M with entries $|$ and $-$ (or $+1$ and -1, respectively) supplied with a collection of $m+n-2$ positive real numbers $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$. The matrix M is called the *weaving matrix*, and the numbers $a_1, \dots, b_{n-1}$ are the *distances* between the rows and columns. On the plane let us draw m parallel lines $k_1, \dots, k_m$ at the distances $a_1, \dots, a_{m-1}$ from each other and n perpendicular to them lines $l_1, \dots, l_n$ at the distances $b_1, \dots, b_{n-1}$ from each other. Now near the intersection point of lines k_i and l_j, let us erase part of the line k_i if $M_{ij} = |$ and part of the line l_j if $M_{ij} = -$ (thus forming "under-crossings" and "over-crossings").

For example, to the matrix

$$A := \begin{pmatrix} - & | & - \\ | & - & | \\ - & | & - \end{pmatrix}$$ corresponds the picture



A weaving is called *realizable* if there exists a family of lines in the three-dimensional space that projects on the drawing constructed from the weaving as above. A weaving is called *non-realizable* if it is not realizable.

In what follows, unless specified otherwise, by a matrix we mean a matrix with entries $|$ and $-$.

In the following problems, let us agree that for any assertion formulated as a problem, a proof of this assertion is required.

Introductory problems

Problem 1. (a) Any $2 \times n$ weaving is realizable (if you can't prove this, see Problem 3 (a) below).

(b) Any weaving with matrix A is non-realizable (if you can't prove this, see Problem 4 (a) below).

Problem 2. (a) The realizability of a weaving does not change if all the entries of its matrix are changed to the opposite ones ($|$ to $-$ and vice versa) and the distances are left unchanged.

Such a transformation of a weaving W is called a *mirror symmetry* and denoted by $W \mapsto SW$.

(b) The realizability of a weaving W does not change under its symmetry in the main diagonal (the rows becoming columns and vice versa) with the distances $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$ replaced by $b_1, \dots, b_{n-1}, a_1, \dots, a_{m-1}$.

Such a transformation of a weaving W is called a *transposition* and denoted by $W \mapsto TW$.

(c) Define the *symmetry of a weaving with respect to horizontal and vertical axes* and prove that this transformation does not affect realizability.

(d) The realizability of an arbitrary weaving does not change if all the distances $a_1, \dots, a_{m-1}$ are multiplied by the same positive real number.

Problem 3. A row (or a column) of the matrix of a weaving is called *monotone* if it begins with a sequence of $|$'s which can be followed only by a sequence of $-$'s or vice versa (no more than one change from $|$ to $-$, or vice versa, is allowed). In particular, if a row (column) contains only $|$'s or only $-$'s, then it is monotone.

(a) The realizability of a weaving does not change if a monotone row (or column) is deleted and the distances are changed accordingly.

(b) A non-monotone row of length 3 is either $(| - |)$, or $(- | -)$.

(c) Among the matrices of 3×3 weavings, only A and SA have no monotone rows.

(d) Realizability is *hereditary*, i.e., any *sub-weaving* (give the definition!) of a realizable weaving is realizable.

(e) A $3 \times n$ weaving is realizable if and only if its matrix contains no minors of the form A and SA.

Problem 4 (The line rotation lemma.). Let us define a *singular* $m \times n$ *weaving* as an $m \times n$ matrix with entries $|, -, +$ (or $+1$, -1 and 0) supplied with a collection of $m+n-2$ positive real numbers $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$. Draw lines k_i and l_j on the plane analogously to the above. Now near the intersection point of lines k_i and l_j, let us erase part of the line k_i if $M_{ij} = |$ and part of the line l_j if $M_{ij} = -$ (and leave the intersection points if $M_{ij} = +$).

The realizability of a singular weaving is defined similarly to that of an ordinary (i.e., non-singular) weaving.

(a) The realizability of a singular $3 \times n$ weaving does not change if a row (column) of the form $(| - |)$ is replaced by $(+ - +)$ and a row (column) of the form $(-|-)$ by $(+|+)$.

(b) The realizability of a singular $4 \times n$ weaving does not change under the following replacements of rows (columns):

$$a = (- - | -),\ b = (-| - -),\ c = (-| \, |-),\ d = (-| - |) \quad \text{by}$$

$$(- - ++),\ (+ + --),\ (+| \, |+),\ (- + +|).$$

(c) State and prove the general property of invariance of realizability under "rotations of lines in space".

(d) If a singular weaving is realizable, then for any two of the parallel lines of the corresponding drawing the realization can be chosen so that the corresponding lines in space are actually parallel (i.e., lie in one plane). (You may use the result of this problem further without proof.)

Problem 5. In this "physical" problem we do not expect rigorous solutions or precise physical models. What is important is to understand the experimental interpretation and use it to perform practical experiments to check conjectures.

(a) Suppose we have a sufficient number of straight but slightly flexible rods (e.g. knitting needles or thin strips of wood). Using them, we can try to construct a model of a given weaving, eventually slightly bending the rods in the process. Show that if the construction is rigid (i.e., does not fall apart), then the given weaving was non-realizable. (And so can be used to weave baskets and other objects.)

(b) A weaving is realizable if and only if the construction described in (a) does not contain any rigid sub-constructions.

Problem 6. (a) All non-monotone rows of length 4 are of the form $a, b, c, d, Sa, Sb, Sc, Sd$ (see Problem 4 (b) for the definition of the rows $a, b, \dots$). Thus any matrix of a $n \times 4$ weaving without monotone rows is determined by an n-letter word in those eight letters.

(b) Which of these words of length n are "forbidden" by the condition that there are no minors of the form A и SA?

(c) Which of these words of length n are "forbidden" by the condition that there are no horizontal monotone lines?

Problem 7. Given two weavings with the same matrix, can it happen that one is realizable and the other is not?

Problem 8. (a) The matrix

$$B = \begin{pmatrix} | & - & - & | \\ - & - & | & - \\ - & | & - & | \\ | & - & | & | \end{pmatrix}$$

contain no monotone rows or columns, nor any minors (=sub-matrices) of the form A, SA.

(b) Any weaving with matrix B is non-realizable.

Problem 9. Find an algorithm determining the realizability of any weaving.

You can use the algorithm from this problem for (computer) experiments concerning realizability of weavings (e.g. those constructed by using Problem 6). Here you can employ standard software for determining the solvability of linear algebraic systems.