NEW WAYS OF WEAVING BASKETS

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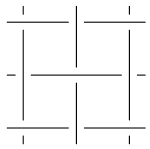
Denn das Leben ist keine Rechnung und keine
mathematische Figur, sondern ein Wunder.
H. Hesse, Kurgast.

Definitions

An $(m \times n)$ *weaving* is defined as an $(m \times n)$ matrix M with entries $|$ and $-$ (or $+1$ and -1, respectively) supplied with a collection of $m + n - 2$ positive real numbers $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$. The matrix M is called the *weaving matrix*, and the numbers $a_1, \dots, b_{n-1}$ are the *distances* between the rows and columns. On the plane let us draw m parallel lines $k_1, \dots, k_m$ at the distances $a_1, \dots, a_{m-1}$ from each other and n perpendicular to them lines $l_1, \dots, l_n$ at the distances $b_1, \dots, b_{n-1}$ from each other. Now near the intersection point of lines k_i and l_j, let us erase part of the line k_i if $M_{ij} = |$ and part of the line l_j if $M_{ij} = -$ (thus forming "under-crossings" and "over-crossings").

For example, to the matrix

$$A := \begin{pmatrix} - & | & - \\ | & - & | \\ - & | & - \end{pmatrix}$$ corresponds the picture



A weaving is called *realizable* if there exists a family of lines in the three-dimensional space that projects on the drawing constructed from the weaving as above. A weaving is called *non-realizable* if it is not realizable.

In what follows, unless specified otherwise, by a matrix we mean a matrix with entries $|$ and $-$.

In the following problems, let us agree that for any assertion formulated as a problem, a proof of this assertion is required.

Introductory problems

Problem 1. (a) Any $2 \times n$ weaving is realizable (if you can't prove this, see Problem 3 (a) below).

(b) Any weaving with matrix A is non-realizable (if you can't prove this, see Problem 4 (a) below).

Problem 2. (a) The realizability of a weaving does not change if all the entries of its matrix are changed to the opposite ones ($|$ to $-$ and vice versa) and the distances are left unchanged.

Such a transformation of a weaving W is called a *mirror symmetry* and denoted by $W \mapsto SW$.

(b) The realizability of a weaving W does not change under its symmetry in the main diagonal (the rows becoming columns and vice versa) with the distances $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$ replaced by $b_1, \dots, b_{n-1}, a_1, \dots, a_{m-1}$.

Such a transformation of a weaving W is called a *transposition* and denoted by $W \mapsto TW$.

(c) Define the *symmetry of a weaving with respect to horizontal and vertical axes* and prove that this transformation does not affect realizability.

(d) The realizability of an arbitrary weaving does not change if all the distances $a_1, \dots, a_{m-1}$ are multiplied by the same positive real number.

Problem 3. A row (or a column) of the matrix of a weaving is called *monotone* if it begins with a sequence of $|$'s which can be followed only by a sequence of $-$'s or vice versa (no more than one change from $|$ to $-$, or vice versa, is allowed). In particular, if a row (column) contains only $|$'s or only $-$'s, then it is monotone.

(a) The realizability of a weaving does not change if a monotone row (or column) is deleted and the distances are changed accordingly.

(b) A non-monotone row of length 3 is either $(| - |)$, or $(- | -)$.

(c) Among the matrices of 3×3 weavings, only A and SA have no monotone rows.

(d) Realizability is *hereditary*, i.e., any *sub-weaving* (give the definition!) of a realizable weaving is realizable.

(e) A $3 \times n$ weaving is realizable if and only if its matrix contains no minors of the form A and SA.

Problem 4 (The line rotation lemma.). Let us define a *singular $m \times n$ weaving* as an $m \times n$ matrix with entries $|$, $-$, $+$ (or $+1$, -1 and 0) supplied with a collection of $m+n-2$ positive real numbers $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$. Draw lines k_i and l_j on the plane analogously to the above. Now near the intersection point of lines k_i and l_j, let us erase part of the line k_i if $M_{ij} = |$ and part of the line l_j if $M_{ij} = -$ (and leave the intersection points if $M_{ij} = +$).

The realizability of a singular weaving is defined similarly to that of an ordinary (i.e., non-singular) weaving.

(a) The realizability of a singular $3 \times n$ weaving does not change if a row (column) of the form $(| - |)$ is replaced by $(+ - +)$ and a row (column) of the form $(-|-)$ by $(+|+)$.

(b) The realizability of a singular $4 \times n$ weaving does not change under the following replacements of rows (columns):

$$a = (- - |-),\ b = (-| - -),\ c = (-| \, |-),\ d = (-| - |) \quad \text{by}$$

$$(- - ++),\ (+ + - -),\ (+| \, |+),\ (- + +|).$$

(c) State and prove the general property of invariance of realizability under "rotations of lines in space".

(d) If a singular weaving is realizable, then for any two of the parallel lines of the corresponding drawing the realization can be chosen so that the corresponding lines in space are actually parallel (i.e., lie in one plane). (You may use the result of this problem further without proof.)

Problem 5. In this "physical" problem we do not expect rigorous solutions or precise physical models. What is important is to understand the experimental interpretation and use it to perform practical experiments to check conjectures.

(a) Suppose we have a sufficient number of straight but slightly flexible rods (e.g. knitting needles or thin strips of wood). Using them, we can try to construct a model of a given weaving, eventually slightly bending the rods in the process. Show that if the construction is rigid (i.e., does not fall apart), then the given weaving was non-realizable. (And so can be used to weave baskets and other objects.)

(b) A weaving is realizable if and only if the construction described in (a) does not contain any rigid sub-constructions.

Problem 6. (a) All non-monotone rows of length 4 are of the form $a, b, c, d, Sa, Sb, Sc, Sd$ (see Problem 4 (b) for the definition of the rows $a, b, \dots$). Thus any matrix of a $n \times 4$ weaving without monotone rows is determined by an n-letter word in those eight letters.

(b) Which of these words of length n are "forbidden" by the condition that there are no minors of the form A и SA?

(c) Which of these words of length n are "forbidden" by the condition that there are no horizontal monotone lines?

Problem 7. Given two weavings with the same matrix, can it happen that one is realizable and the other is not?

Problem 8. (a) The matrix

$$B = \begin{pmatrix} | & - & - & | \\ - & - & | & - \\ - & | & - & | \\ | & - & | & | \end{pmatrix}$$

contain no monotone rows or columns, nor any minors (=sub-matrices) of the form A, SA.

(b) Any weaving with matrix B is non-realizable.

Problem 9. Find an algorithm determining the realizability of any weaving.

You can use the algorithm from this problem for (computer) experiments concerning realizability of weavings (e.g. those constructed by using Problem 6). Here you can employ standard software for determining the solvability of linear algebraic systems.

Problems proposed for the second half of the Conference

A matrix is called *realizable* if it can be supplied with distances so as to give a realizable weaving.

Problem 10. (a) The *saddle-like transformation* $(x, y, z) \mapsto (x, y, z + xy)$ takes lines perpendicular to the x-axis or to the y-axis to straight lines. (This can be used to solve Problem 4 (d).)

(b) What is the image of the plane $z = const$ under the saddle-like transformation?

Problem 11. Suppose that two parallel lines k and k' are chosen in some (singular) weaving, and all the other lines are supplied with orientations. We are allowed to move along the lines k and k' in any direction, along the other lines in the direction of their orientation, and jump from one line to another one if the first line passes "over" the other one at that point or intersects it at that point. We shall say that the weaving is *acyclic* if for any pair of parallel lines we can choose orientations on all the other lines so that it is impossible to move in the weaving (in accordance to the rules above) and return to the starting point.

(a) Two (singular) weavings with the same matrix are either both acyclic or both non-acyclic.

(b) Any realizable matrix is acyclic. (The result of this problem can be used to prove the non-realizability of the matrices A and B.)

(c)* Is it true that any acyclic matrix can be realized?

(d) In the definition of acyclicity, let us choose, instead of the two parallel lines k and k', two *perpendicular lines* k and l. Solve similar problems for this definition of acyclicity.

(e) In the definition of acyclicity, let us choose two parallel lines k and k' and one more line l perpendicular to them. Solve similar problems for this definition of acyclicity.

Problem 12. (a) The realizability of a weaving with matrix

$$B' = \begin{pmatrix} | & - & | & | \\ - & - & | & - \\ | & | & - & | \\ | & - & | & | \end{pmatrix}$$

is equivalent to the realizability of a weaving with the same distances and matrix

$$\begin{pmatrix} + & - & | & + \\ + & - & | & + \\ | & + & + & | \\ | & + & + & | \end{pmatrix}$$

(b) If a_1 is very small comparatively to the other a_i's and b_j's, then the given weaving is realizable.

(c) If $a_2 = b_2 \ll a_1 = a_3 = b_1 = b_3$, then the given weaving is non-realizable.

Problem 13 (Projective transformations of matrices). (a) The realizability of a matrix does not change if we place the first column of the matrix right after the last one and simultaneously change all the column's entries to opposite ones. Realizability does not change under similar transformations of rows. Such transformations are denoted by X and Y and called *projective transformations* of matrices. (You can use the result of this problem in further problems without proof.)

(b) $XA = YA = SA$.

(c) The property of rows and columns to be monotone does not change under projective transformations.

(d) $S = X^m = Y^n$.

(e) $XY = YX$.

Problem 14. (a) All the letters $a, \ldots, Sd$ from Problem 6 (a) can be obtained from one of them by projective transformations. Thus any $4 \times n$ matrix without monotone rows or columns can be specified by a word of length n in the numbers $1, 2, \ldots, 8$.

(b) Any $4{\times}4$ matrix without monotone rows or columns not containing the sub-matrices A, SA can be obtained from B or from B' by a composition of mirror symmetries, transpositions, axial symmetries and projective transformations.

(c) What words are forbidden by the condition that there are no sub-matrices the form A, SA, B, X^pY^qB?

(d) What words remain?

(e) Which of them are realizable?

Problem 15. (a) Does there exist infinitely many non-realizable matrices all of whose sub-matrices are realizable (compare [Kva89, RSS04])?

(b) Does there exist an infinite collection of realizable matrices none of which contain a monotone row or column?

(c) * Does there exist an infinite collection of realizable matrices none of which contains a monotone row or column or any submatrix from the family?

Problem 16. (a) Find the number of non-realizable 4×4 matrices.

(b) Find all the matrices 4×5 without monotone rows or columns and non-realizable sub-matrices up to symmetries, transposition and projective transformations (you can develop and use computer program for this problem).

Problem 17. (a) A matrix is called *strongly realizable* if *for any* collection of distances it determines a realizable weaving. Prove analogs of the above problems for strong realizability.

Problem 18. Let $\mathtt{W}$ denote the set of all $m \times n$ matrices up to all symmetries, transposition, and projective transformations.

(a)* Find (or estimate) the number of elements in $\mathtt{W}$.

(b)* Find (or estimate) the ratio of realizable elements of $\mathtt{W}$ to non realizable ones.

(c)* Find (or estimate) the ratio of strongly realizable elements of $\mathtt{W}$ to non strongly realizable ones.

The authors assume that participants should not try to solve all the suggested problems during the conference. Please try to solve the most interesting ones, and return to the other problems later.

Solutions and Hints (Introductory Problems)

Problem 1. (a) Since any line whose projection intersects the parallel projections of two lines is necessarily monotone, the statement of the problem follows from Problem 3 (a).

(b) Suppose that some weaving with matrix A is realizable. Let us transform the matrix A by using the lemma on the rotation of lines (Problem 4 (a)):

$$\begin{pmatrix} - & | & - \\ | & - & | \\ - & | & - \end{pmatrix} \mapsto \begin{pmatrix} + & | & + \\ | & - & | \\ + & | & + \end{pmatrix} \mapsto \begin{pmatrix} + & + & + \\ + & - & + \\ + & + & + \end{pmatrix}.$$

According to the lemma, the matrices obtained are also realizable. Let us consider a configuration of lines in space realizing the last matrix. Let $\widetilde{A}_{ij}$ be the intersection point of the lines $\widetilde{k}_i$ and $\widetilde{l}_j$ (for $(i,j) \neq (2,2)$). Consider the tetrahedron $\widetilde{A}_{11}\widetilde{A}_{13}\widetilde{A}_{31}\widetilde{A}_{33}$. The lines $\widetilde{k}_2$ and $\widetilde{l}_2$ divide its opposite edges in equal ratios; therefore, they intersect, which contradicts the realizability of the matrix under consideration. The contradiction obtained shows that any weaving with matrix A is non-realizable. $\square$

Problem 2. (a) Suppose that the lines $\widetilde{k}_i$, $\widetilde{l}_j$ realize some matrix U. Then the lines symmetric to the lines $\widetilde{k}_i$, $\widetilde{l}_j$ with respect to the plane Oxy will realize the matrix SU. The solutions of items (b) and (c) are similar.

(d) Choose a coordinate system in the three-dimensional space so that Ox is parallel to the horizontal lines, Oy to the vertical lines and Oz is directed upwards. Let $r_1, \dots, r_{m+n}$ be a collection of lines realizing some weaving, where $r_i = \{(x_i^{(0)} + x_i^{(1)}t, y_i^{(0)} + y_i^{(1)}t, z_i^{(0)} + z_i^{(1)}t)\}$. Then the collection of lines $r_i' = \{(x_i^{(0)} + x_i^{(1)}t, py_i^{(0)} + py_i^{(1)}t, z_i^{(0)} + z_i^{(1)}t)\}$ realizes the weaving obtained from the given one by multiplying all the distances $a_1, \dots, a_{m-1}$ by p. $\square$

Problem 3. (a) Let P be a weaving with a monotone row (or column) and let P' be the weaving obtained from it by deleting this line. Obviously, if P is realizable, then so is P'. Let us prove the opposite implication. Suppose that P' is realizable. Consider a corresponding configuration of lines in space. To the projection of these lines on the xy-plane, let us add the monotone line of P; on this line, let us mark a point to one side of which this line has only overpasses and to the other side of which it has only underpasses. Constructing a vertical line through this point and inclining it slightly, we will obtain a configuration of lines which realizes P.

(b) It suffices to check all 8 rows of length 3.

(c) Each row of the matrix not containing monotone lines must be of the form $(|-|)$ or $(-|-)$. If the middle row coincides with one of the other rows, then all the columns are monotone. Thus the matrix under consideration is either A or SA.

(d) This is obvious: in order to realize the sub-weaving, it suffices to take the corresponding lines of the realization of the weaving.

(e) Consider a $3 \times n$ weaving W. By item (a), we can omit all monotone columns from our considerations. As the result only columns of the form $\begin{pmatrix} | \\ - \\ | \end{pmatrix}$ и $\begin{pmatrix} - \\ | \\ - \end{pmatrix}$. will remain. If W is non-realizable, then it must have a non-monotone row. Therefore there are three columns whose forms alternate. These columns form the sub-matrix A or the sub-matrix SA. Now if the matrix of our weaving has such a sub-matrix, then W is non-realizable because it has a non-realizable sub-matrix and realizability is hereditary. $\square$

Problem 4. (a) Because of mirror symmetry, it suffices to prove the first part of the statement. Let us prove that the realizability of the weaving with j-th line $(|-|)$ implies the realizability of the weaving obtained from the given one by replacing j-th line $(+-+)$. Let $\widetilde{B}$ be the point on the line $\widetilde{k}_j$ which projects on the point B. Let us rotate the line $\widetilde{k}_j$ about the point $\widetilde{B}$ until it intersects the line $\widetilde{l}_1$ at the point $\widetilde{A}_{1j}$. After that rotate the line $\widetilde{k}_j$ about $\widetilde{A}_{1j}$ until it intersects $\widetilde{l}_3$. The result will be a configuration of points realizing the required weaving (with row $(+-+)$).

(b) The solution is similar to that of the previous problem.

(d) Use the saddle-like transformation. $\square$

Problem 7. This follows from problem 12. $\square$

Problem 8. (a) This follows from a direct verification. $\square$

Problem 9. Each line of the configuration realizing the weaving is given by the z-coordinate of one of its points and its slope (i.e., the tangent of its angle) with respect to the xy-plane. Note that in these variables the realizability condition of the equivalent to the solvability of a system of linear equations and inequalities (one for each of the crossings of the projection of the weaving). Therefore, to check realizability it suffices to verify that the system has at least one solution. For an algorithm for solving systems of linear equations and inequalities, see [Sol].

$\square$