NEW WAYS OF WEAVING BASKETS

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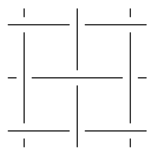
Denn das Leben ist keine Rechnung und keine mathematische Figur, sondern ein Wunder.
H. Hesse, Kurgast.

Definitions

An $(m \times n)$ *weaving* is defined as an $(m \times n)$ matrix M with entries $|$ and $-$ (or $+1$ and -1, respectively) supplied with a collection of $m+n-2$ positive real numbers $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$. The matrix M is called the *weaving matrix*, and the numbers $a_1, \dots, b_{n-1}$ are the *distances* between the rows and columns. On the plane let us draw m parallel lines $k_1, \dots, k_m$ at the distances $a_1, \dots, a_{m-1}$ from each other and n perpendicular to them lines $l_1, \dots, l_n$ at the distances $b_1, \dots, b_{n-1}$ from each other. Now near the intersection point of lines k_i and l_j, let us erase part of the line k_i if $M_{ij} = |$ and part of the line l_j if $M_{ij} = -$ (thus forming "under-crossings" and "over-crossings").

For example, to the matrix

$$A := \begin{pmatrix} - & | & - \\ | & - & | \\ - & | & - \end{pmatrix}$$ corresponds the picture 

A weaving is called *realizable* if there exists a family of lines in the three-dimensional space that projects on the drawing constructed from the weaving as above. A weaving is called *non-realizable* if it is not realizable.

In what follows, unless specified otherwise, by a matrix we mean a matrix with entries $|$ and $-$.

In the following problems, let us agree that for any assertion formulated as a problem, a proof of this assertion is required.

Introductory problems

Problem 1. (a) Any $2 \times n$ weaving is realizable (if you can't prove this, see Problem 3 (a) below).

(b) Any weaving with matrix A is non-realizable (if you can't prove this, see Problem 4 (a) below).

Problem 2. (a) The realizability of a weaving does not change if all the entries of its matrix are changed to the opposite ones ($|$ to $-$ and vice versa) and the distances are left unchanged.

Such a transformation of a weaving W is called a *mirror symmetry* and denoted by $W \mapsto SW$.

(b) The realizability of a weaving W does not change under its symmetry in the main diagonal (the rows becoming columns and vice versa) with the distances $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$ replaced by $b_1, \dots, b_{n-1}, a_1, \dots, a_{m-1}$.

Such a transformation of a weaving W is called a *transposition* and denoted by $W \mapsto TW$.

(c) Define the *symmetry of a weaving with respect to horizontal and vertical axes* and prove that this transformation does not affect realizability.

(d) The realizability of an arbitrary weaving does not change if all the distances $a_1, \dots, a_{m-1}$ are multiplied by the same positive real number.

Problem 3. A row (or a column) of the matrix of a weaving is called *monotone* if it begins with a sequence of $|$'s which can be followed only by a sequence of $-$'s or vice versa (no more than one change from $|$ to $-$, or vice versa, is allowed). In particular, if a row (column) contains only $|$'s or only $-$'s, then it is monotone.

(a) The realizability of a weaving does not change if a monotone row (or column) is deleted and the distances are changed accordingly.

(b) A non-monotone row of length 3 is either $(| - |)$, or $(-|-)$.

(c) Among the matrices of 3×3 weavings, only A and SA have no monotone rows.

(d) Realizability is *hereditary*, i.e., any *sub-weaving* (give the definition!) of a realizable weaving is realizable.

(e) A $3 \times n$ weaving is realizable if and only if its matrix contains no minors of the form A and SA.

Problem 4 (The line rotation lemma.). Let us define a *singular $m \times n$ weaving* as an $m \times n$ matrix with entries $|, -, +$ (or $+1$, -1 and 0) supplied with a collection of $m+n-2$ positive real numbers $a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}$. Draw lines k_i and l_j on the plane analogously to the above. Now near the intersection point of lines k_i and l_j, let us erase part of the line k_i if $M_{ij} = |$ and part of the line l_j if $M_{ij} = -$ (and leave the intersection points if $M_{ij} = +$).

The realizability of a singular weaving is defined similarly to that of an ordinary (i.e., non-singular) weaving.

(a) The realizability of a singular $3 \times n$ weaving does not change if a row (column) of the form $(| - |)$ is replaced by $(+ - +)$ and a row (column) of the form $(-|-)$ by $(+|+)$.

(b) The realizability of a singular $4 \times n$ weaving does not change under the following replacements of rows (columns):

$$a = (- - | -),\ b = (-| - -),\ c = (-| \, | -),\ d = (-| - |) \quad \text{by}$$

$$(- - + +),\ (+ + - -),\ (+| \, | +),\ (- + + |).$$

(c) State and prove the general property of invariance of realizability under "rotations of lines in space".

(d) If a singular weaving is realizable, then for any two of the parallel lines of the corresponding drawing the realization can be chosen so that the corresponding lines in space are actually parallel (i.e., lie in one plane). (You may use the result of this problem further without proof.)

Problem 5. In this "physical" problem we do not expect rigorous solutions or precise physical models. What is important is to understand the experimental interpretation and use it to perform practical experiments to check conjectures.

(a) Suppose we have a sufficient number of straight but slightly flexible rods (e.g. knitting needles or thin strips of wood). Using them, we can try to construct a model of a given weaving, eventually slightly bending the rods in the process. Show that if the construction is rigid (i.e., does not fall apart), then the given weaving was non-realizable. (And so can be used to weave baskets and other objects.)

(b) A weaving is realizable if and only if the construction described in (a) does not contain any rigid sub-constructions.

Problem 6. (a) All non-monotone rows of length 4 are of the form $a, b, c, d, Sa, Sb, Sc, Sd$ (see Problem 4 (b) for the definition of the rows $a, b, \dots$). Thus any matrix of a $n \times 4$ weaving without monotone rows is determined by an n-letter word in those eight letters.

(b) Which of these words of length n are "forbidden" by the condition that there are no minors of the form A и SA?

(c) Which of these words of length n are "forbidden" by the condition that there are no horizontal monotone lines?

Problem 7. Given two weavings with the same matrix, can it happen that one is realizable and the other is not?

Problem 8. (a) The matrix

$$B = \begin{pmatrix} | & - & - & | \\ - & - & | & - \\ - & | & - & | \\ | & - & | & | \end{pmatrix}$$

contain no monotone rows or columns, nor any minors (=sub-matrices) of the form A, SA.

(b) Any weaving with matrix B is non-realizable.

Problem 9. Find an algorithm determining the realizability of any weaving.

You can use the algorithm from this problem for (computer) experiments concerning realizability of weavings (e.g. those constructed by using Problem 6). Here you can employ standard software for determining the solvability of linear algebraic systems.

Problems proposed for the second half of the Conference

A matrix is called *realizable* if it can be supplied with distances so as to give a realizable weaving.

Problem 10. (a) The *saddle-like transformation* $(x, y, z) \mapsto (x, y, z + xy)$ takes lines perpendicular to the x-axis or to the y-axis to straight lines. (This can be used to solve Problem 4 (d).)

(b) What is the image of the plane $z = const$ under the saddle-like transformation?

Problem 11. Suppose that two parallel lines k and k' are chosen in some (singular) weaving, and all the other lines are supplied with orientations. We are allowed to move along the lines k and k' in any direction, along the other lines in the direction of their orientation, and jump from one line to another one if the first line passes "over" the other one at that point or intersects it at that point. We shall say that the weaving is *acyclic* if for any pair of parallel lines we can choose orientations on all the other lines so that it is impossible to move in the weaving (in accordance to the rules above) and return to the starting point.

(a) Two (singular) weavings with the same matrix are either both acyclic or both non-acyclic.

(b) Any realizable matrix is acyclic. (The result of this problem can be used to prove the non-realizability of the matrices A and B.)

(c)* Is it true that any acyclic matrix can be realized?

(d) In the definition of acyclicity, let us choose, instead of the two parallel lines k and k', two *perpendicular lines* k and l. Solve similar problems for this definition of acyclicity.

(e) In the definition of acyclicity, let us choose two parallel lines k and k' and one more line l perpendicular to them. Solve similar problems for this definition of acyclicity.

Problem 12. (a) The realizability of a weaving with matrix

$$B' = \begin{pmatrix} | & - & | & | \\ - & - & | & - \\ | & | & - & | \\ | & - & | & | \end{pmatrix}$$

is equivalent to the realizability of a weaving with the same distances and matrix

$$\begin{pmatrix} + & - & | & + \\ + & - & | & + \\ | & + & + & | \\ | & + & + & | \end{pmatrix}$$

(b) If a_1 is very small comparatively to the other a_i's and b_j's, then the given weaving is realizable.

(c) If $a_2 = b_2 \ll a_1 = a_3 = b_1 = b_3$, then the given weaving is non-realizable.

Problem 13 (Projective transformations of matrices). (a) The realizability of a matrix does not change if we place the first column of the matrix right after the last one and simultaneously change all the column's entries to opposite ones. Realizability does not change under similar transformations of rows. Such transformations are denoted by X and Y and called *projective transformations* of matrices. (You can use the result of this problem in further problems without proof.)

(b) $XA = YA = SA$.

(c) The property of rows and columns to be monotone does not change under projective transformations.

(d) $S = X^m = Y^n$.

(e) $XY = YX$.

Problem 14. (a) All the letters $a, \dots, Sd$ from Problem 6 (a) can be obtained from one of them by projective transformations. Thus any $4 \times n$ matrix without monotone rows or columns can be specified by a word of length n in the numbers $1, 2, \dots, 8$.

(b) Any 4×4 matrix without monotone rows or columns not containing the sub-matrices A, SA can be obtained from B or from B' by a composition of mirror symmetries, transpositions, axial symmetries and projective transformations.

(c) What words are forbidden by the condition that there are no sub-matrices the form A, SA, B, X^pY^qB?

(d) What words remain?

(e) Which of them are realizable?

Problem 15. (a) Does there exist infinitely many non-realizable matrices all of whose sub-matrices are realizable (compare [Kva89, RSS04])?

(b) Does there exist an infinite collection of realizable matrices none of which contain a monotone row or column?

(c) * Does there exist an infinite collection of realizable matrices none of which contains a monotone row or column or any submatrix from the family?

Problem 16. (a) Find the number of non-realizable 4×4 matrices.

(b) Find all the matrices 4×5 without monotone rows or columns and non-realizable sub-matrices up to symmetries, transposition and projective transformations (you can develop and use computer program for this problem).

Problem 17. (a) A matrix is called *strongly realizable* if *for any* collection of distances it determines a realizable weaving. Prove analogs of the above problems for strong realizability.

Problem 18. Let `W` denote the set of all $m \times n$ matrices up to all symmetries, transposition, and projective transformations.

(a)* Find (or estimate) the number of elements in `W`.

(b)* Find (or estimate) the ratio of realizable elements of `W` to non realizable ones.

(c)* Find (or estimate) the ratio of strongly realizable elements of `W` to non strongly realizable ones.

The authors assume that participants should not try to solve all the suggested problems during the conference. Please try to solve the most interesting ones, and return to the other problems later.

Solutions and hints presented after the final

If in the sequel we consider a coordinate system in the 3-space, then we assume that Ox is parallel to the horizontal lines of the weaving, Oy to the vertical and Oz is directed upwards.

Problem 4. (d) Let us make the lines $\widetilde{k}_i$ and $\widetilde{k}_j$ parallel. Assume that the line $\widetilde{k}_s$ is presented by equation $(x_s + t, y_s, z_s + q_s t)$. If $q_i = q_j$, then we are done. Now assume that $q_i \neq q_j$. Making a saddle-like transformation (see problem 10 (a)) we obtain another set of lines realizing our weaving. Let us choose a coordinate system so that images of $\widetilde{k}_i$ and $\widetilde{k}_j$ under saddle-like transformation would be parallel. These images have the form $(x_i + t, y_i, z_i + x_i y_i + (q_i + y_i)t)$. They will be parallel if $q_i + y_i = q_j + y_j$, i.e. for $q_i - q_j = y_j - y_i$. Since the left part is non-zero, we can choose a scale on the Oy-axis so that the equality would hold. □

Problem 6. (a) Evident.

(b) We shall use projective transformations (see problem 13). Consider three rows containing minor A or SA. By projective transformation we may assume that this minor will consusts of the first three columns and will be A. Then clearly the first row is b or d, the second is Sb or Sd and the third is b or d. Therefore any forbidden word is obtained from one of these eight words by projective transformation. In the notation of problem 14 (a) these words are 737, 732, 767, 762, 237, 232, 267, 262. A projective transformation adds (mod 8) the unity to each letter of such a word. Then each of these words corresponds to eight forbidden words, while the words 737 and 262 give the same collections of eight words. So there are 56 forbidden words.

(c) The first column is monotone if first we have only words a, b, c, d in any order, and then only words Sa, Sb, Sc, Sd in any order (or vise versa). For other columns the answer is analogous. □

Problem 10. (a) It suffices to prove that a saddle-like transformation carries straight lines perpendicular to the x-axis to straight lines. This follows because in any plane $x = const$ we have an affine transformaton.

(b) Plane $z = const$ goes to the surface $z - xy = const$, i.e. to a parabolic hyperboloid. These surfaces have the form of a saddle. □

Problem 11. (a) Evident.

(b) Consider a realizable weaving and lines realizing this weaving. Using saddle-like transformation we can make lines $\widetilde{k}$ and $\widetilde{k}'$ are parallel. Then by a linear transformation $x' = x$, $y' = y$, $z' = z - px$ we can make $\widetilde{k}$ and $\widetilde{k}'$ parallel to the Oxy plane. Now by a linear transformation $x' = x$, $y' = y$, $z' = z - qy$ we can make also $\widetilde{l}$ parallel to the Oxy plane. Now orient all the lines except k, k' and l by decreasing of the z-coordinate (if some line is parallel to the plane Oxy then orient this line arbitrarily). When we move around any cycle according to the rules, we only move down. This is impossible because at the end we return to the starting point. Contradiction. □

Problem 12. (a) It suffices to use the line rotation lemma.

(b) Recall that a_i are distances between *horizontal* lines. It suffices to prove realizability of the given weaving with the matrix changed to that obtained in (a). The matrix obtained by deleting the first row is realizable because all its columns are monotone. Take a set of lines realizing the matrix and add a line close to $\widetilde{k}_2$ and intersecting $\widetilde{l}_1$ and $\widetilde{l}_4$. Then the types of intersections of the added line with other lines will be the same as that $\widetilde{k}_2$. Hence we have obtained a set of lines realizing the matrix B'.

(c) It suffices to prove the non-realizability of the given weaving with the matrix changed to that obtained in (a). By the lemma on parallel lines (problem 4 (d)) we may assume that $\widetilde{k}_3$ and $\widetilde{k}_4$ are parallel. By a linear transformation we can obtain that both these lines lie in plane Oxy.

Assume that $a_1 = 1$ and let $\varepsilon = a_2$. Let d be the difference of z-coordinates of points of $\widetilde{k}_2$ projecting to $k_2 \cap l_2$ and to $k_2 \cap l_3$. Then the slope coefficient of $\widetilde{k}_2$ is d/ε. Hence

$$z(\widetilde{k}_2 \cap \widetilde{l}_1) < \frac{d}{\varepsilon} + d \quad \text{and} \quad z(\widetilde{k}_2 \cap \widetilde{l}_4) < -\frac{d}{\varepsilon}.$$

Since l_4 goes above k_3 and intersects k_2, it follows that the slope coefficient of $\widetilde{l}_4$ does not exceed $\frac{d}{\varepsilon}$. Hence

$$z(\widetilde{l}_4 \cap \widetilde{k}_1) < -\frac{d}{\varepsilon} - \frac{d}{\varepsilon^2}.$$

The slope coefficient of $\widetilde{k}_1$ does not exceed $\frac{d}{\varepsilon^2}$. Hence

$$z(\widetilde{l}_1 \cap \widetilde{k}_1) > \frac{d}{\varepsilon^2}.$$

At last considering $\widetilde{l}_1$ we obtain that the z-coordinate of the point of $\widetilde{l}_1$ projecting to $l_1 \cap k_4$ does not exceed

$$\frac{d}{\varepsilon^2} - \left(\frac{d}{\varepsilon^2} - \frac{d}{\varepsilon} - d\right)(2 + \varepsilon) < 0 \quad \text{for} \quad \varepsilon < 1/3.$$

□

Problem 13. (a) Choose a coordinate system in space so that in the half-space $x \geqslant 0$ would lie only one vertical line. Now using the transformation $(x, y, z) \mapsto (x, -1/y, z/y)$. Other statements of this problem are verified directly. □

Problem 15. (a) Let us prove that the matrix

$$C_n = \begin{pmatrix} \vert & - & - & \dots & - & - & \vert \\ - & \vert & - & \dots & - & - & - \\ \vert & - & \vert & \dots & - & - & - \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vert & \vert & \vert & \dots & - & \vert & - \\ \vert & \vert & \vert & \dots & \vert & - & \vert \end{pmatrix}_{n\times n}$$

for $n \geqslant 3$ is non-realizable and each its submatrix is realizable.

First we prove that it is non-realizable. Analogously to the line rotation lemma it suffices to prove that the following matrix is non-realizable:

$$\begin{pmatrix} + & - & - & \dots & - & - & + \\ + & + & - & \dots & - & - & - \\ \vert & + & + & \dots & - & - & - \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vert & \vert & \vert & \dots & + & + & - \\ \vert & \vert & \vert & \dots & \vert & + & + \end{pmatrix}_{n\times n},$$

Suppose that the matrix is realizable. By saddle-like transformation we may assume that $\widetilde{k}_1$ and $\widetilde{k}_2$ are parallel. By linear transformation we may assume that they lie in the plane Oxy. Let

$$z_i = z(\widetilde{k}_i \cap \widetilde{l}_i) \quad \text{and} \quad z'_i = z(\widetilde{k}_{i+1} \cap \widetilde{l}_i).$$

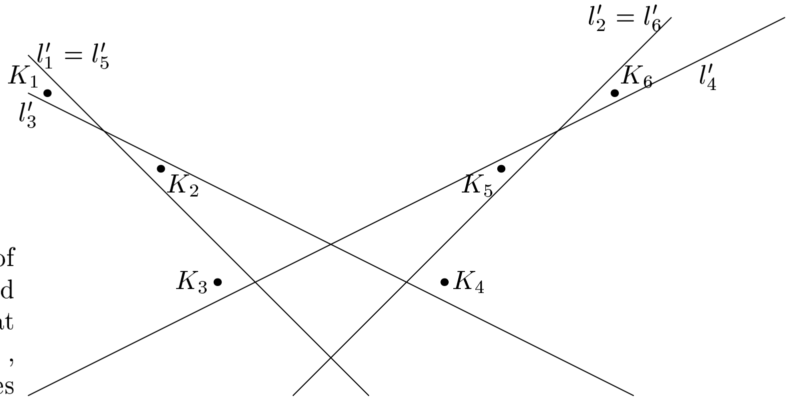
For $i > 1$ we have $z_i < z'_i < z_{i+1}$. Hence $z_n > z_1 = 0$. There remains to note that $\widetilde{l}_n$ first intersects the plane Oxy, then pass under this plane and next over this plane. This is impossible.

Any sub-matrix of C_n is realizable because any its submatrix contains a monotone row.

(b) One such weaving is B'. For any set of lines in space projecting to an $n \times n$ weaving without monotone lines construct such a set with double number of lines. Transfer all lines of the given set by vector (u, u, v), where $u > a_1 + \dots + a_{n-1} + b_1 + \dots + b_{n-1}$ and v is sufficiently large so that over intersection points lines from the transferred set pass over lines of the given set. Then the new weaving does not have monotone lines. □

Problem 17. Let us prove strong realizability of the following matrix containing no monotone rows or columns.

$$\begin{pmatrix} - & \vert & - & \vert & \vert & \vert \\ \vert & \vert & \vert & - & \vert & - \\ \vert & - & - & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert & - & - \\ - & \vert & - & \vert & \vert & \vert \\ \vert & \vert & \vert & - & \vert & - \end{pmatrix}.$$



Suppose that sets of distances between lines of the weaving is given. Then we can draw points and lines on the plane as on the the picture so that the distances between projections of points $K_1, \dots, K_6$ on the horizontal line equal to the distances between the corresponding lines of the weaving.

Now we present the realization. Direct the Ox-axis horizontaly in our plane, the Oz-axis verticaly in our plane and the Oy-axis perpendicular to our plane. Take lines $\widetilde{k}_1, \dots, \widetilde{k}_6$ perpendicular to our plane and intersecting it in corresponding points. Take lines $\widetilde{l}_1, \dots, \widetilde{l}_6$ parallel our plane, at given distances from our plane and projecting to $l'_1, \dots, l'_6$. Clearly this is a realization of our matrix. □