PERSPECTIVE-ORTHOLOGIC TRIANGLES AND TETRAHEDRONS

A.Zaslavsky (with participation of L. and T.Emelyanov)

1 Definitions and introductory problems

Definition 1. Two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are called *perspective*, if the lines A_1A_2, B_1B_2, and C_1C_2 are concurent. The common point of these lines is called *the perspective center*.

Definition 2. Two tetrahedrons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are called *perspective*, if the lines A_1A_2, B_1B_2, C_1C_2 and D_1D_2 are concurent. Their common point is called *the perspective center*.

1. **The Desargue theorem.**

a) Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are perspective if and only if the common points of respective sidelines (A_1B_1 and A_2B_2, B_1C_1 and B_2C_2, C_1A_1 and C_2A_2) are collinerar. The line passing through these points is called *the perspective axis*.

b) Prove that the tetrahedrons $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are perspective if and only if their respective edges intersect and six points of intersection are in the same plane (*the perspective plane*).

Definition 3. The triangle $A_2B_2C_2$ is called *orthologic* to the triangle $A_1B_1C_1$, if the perpendiculars from A_1 to B_2C_2, from B_1 to A_2C_2 and from C_1 to A_2B_2 are concurent. Their common point is called *the orthologic center*.

Definition 4. The tetrahedron $A_2B_2C_2D_2$ is called *orthologic* to the tetrahedron $A_1B_1C_1D_1$, if the perpendiculars from A_1 to the plane $B_2C_2D_2$, from B_1 to $D_2A_2C_2$, from C_1 to $D_2A_2B_2$ and from D_1 to $A_2B_2C_2$ are concurent. Their common point is called *the orthologic center*.

2. **The symmetry of orthology.**

a) Prove that if the triangle $A_2B_2C_2$ is orthologic to the triangle $A_1B_1C_1$ the $A_1B_1C_1$ is orthologic to $A_2B_2C_2$. (The ortologic centers can be different).

b) Prove that the tetrahedron $A_1B_1C_1D_1$ is orthologic to the tetrahedron $A_2B_2C_2D_2$ if and only if their "antirespective" edges (for exemple A_1B_1 and C_2D_2) are perpendicular.

c) Prove that if the tetrahedron $A_1B_1C_1D_1$ is orthologic to the tetrahedron $A_2B_2C_2D_2$ then $A_2B_2C_2D_2$ is orthologic to $A_1B_1C_1D_1$ (the orthologic centers can be different).

In next problems we shall examine the properties of pairs of triangles (tetrahedrons) which are both perspective and orthologic.

3. **Theorem 1.**

a) Let P is the perspective center of two triangles (not homothetic), Q is one of their orthologic centers. Prove that the line PQ is perpendicular to the perspective axis.

b) Let P is the perspective center of two tetrahedrons (not homothetic), Q is one of their orthologic centers. Prove that PQ is perpendicular to the perspective plane.

4. **Theorem 2.**

a) Let P is the perspective center of two triangles, Q_1, Q_2 are their orthologic centers. Prove that P, Q_1, Q_2 are collinear.

b) Let P is the perspective center of two tetrahedrons, Q_1, Q_2 are their orthologic centers. Prove that P, Q_1, Q_2 are collinear.

2 Basic problem

The problems of this part can be solved by using the theorems 1 and 2. We use next designations. O is the circumcenter of the triangle ABC, I is its incenter, H is its orthocenter, G is the Gergonne point (the common point of segments between the vertex of ABC and the touching points of its sidelines with the incircle), N is the Nagel point (the common point of segments between the vertex and the touching points of the sidelines with the excircles), P is symmetric to I with respect to O, Q is isogonally conjugated to P (the lines AP and AQ, BP and BQ, CP and CQ are symmetric with respect to the bissectrix of ABC), L is the Lemoine point (isogonally conjugated to the centroid M).

Definition 5. Let X is an arbitrary point of plane ABC, A', B', C' the common points of AX, BX, CX with opposite sidelines of triangle ABC. Then the perspective axis of triangles ABC and $A'B'C'$ is called *the trypolar* of X with respect to triangle ABC.

5. Prove that the trypolar of G is perpendicular to IG.

6. Prove that the trypolar of H is perpendicular to Eiler line.

7. Prove that P, Q, N are collinear.

8. Prove that the trypolar of N is perpendicular to IH.

9. Let A_1, B_1, C_1 are the midpoints of arcs BC, CA, AB of the circumcircle ABC not containing its vertex, E, F are the common points of AC and A_1C_1, BC and B_1C_1 respectively. Prove that the lines EF and OI are perpendicular.

10. Prove that the common points of external bisectrix with the opposite sidelines of the triangle are collinear and this line is perpendicular to OI.

11. Prove that the common points of the lines touching the circumcircle of the triangle in its vertex with the opposite sidelines are collinear and this line is perpendicular to OL.

3 Additional problems

12. Prove that if two triangles are orthologic and their orthologic centers coincide then the triangles are perspective.

13. Is this fact correct for the tetrahedrons?

14. All tetrahedrons perspective-orthologic to given tetrahedron $ABCD$ are considered. Find the locus of

a) the perspective centers;

b) the orthologic centers.

15. Try to find new corollary of theorems 1 and 2.