PERSPECTIVE-ORTHOLOGIC TRIANGLES AND TETRAHEDRONS
Solutions

1 Definitions and introductory problems

1. **The Desargue theorem.**

a) There are many proofs of Desargue theorem but the most elegant is connected with "the exit to the space". Its idea is the next. The perspective center P is considered as located above the plane $A_1B_1C_1$. Then the common point of A_1B_1 and A_2B_2 belong to both the planes $A_1B_1C_1$ and $A_2B_2C_2$. So it is in the intersection line of these planes. Two another points also are in this line. We propose you to understand why any plain configuration can be considered as the projection of 3-dimensional.

b) As A_1A_2 and B_1B_2 intersect the points A_1, A_2, B_1, B_2 are complanar. So A_1B_1 and A_2B_2 intersect (may be in infinite point). Similarly another pairs of respective edges also intersect. The common points of A_1B_1 and A_2B_2, A_1C_1 and A_2C_2, B_1C_1 and B_2C_2 are in in the intersection line of the planes $A_1B_1C_1$ and $A_2B_2C_2$. This is also correct for the common points of edges belonging to three another pairs of respective faces. Any two of these four lines intersect and the intersectiopn points are different. So the four lines are in the plane.

2. **The symmetry of orthology.**

a) Let M is the projection of B_1 to A_2C_2, N_1, N_2, N_3 are the projections of A_2, B_2, C_2 to respective sideline of $A_1B_1C_1$. As the angles A_2MB_1 and $A_2N_1B_1$ are direct, the points A_2, B_1, M, N_1 are concyclic. So $\sin\angle Q_1B_1C_1 = \sin\angle C_2A_2N_1$. Similarly $\sin\angle Q_1B_1C_1 = \sin\angle A_2C_2N_3$, $\sin\angle Q_1A_1B_1 = \sin\angle B_2C_2N_3$, $\sin\angle Q_1A_1C_1 = \sin\angle C_2B_2N_2$, $\sin\angle Q_1C_1A_1 = \sin\angle A_2B_2N_2$, $\sin\angle Q_1C_1B_1 = \sin\angle B_2A_2N_1$. Now the affirmation of the problem follows from Cheva theorem for the angles.

b) Let P is the common point of perpendiculars from vertex of $A_2B_2C_2D_2$ to the faces of $A_1B_1C_1D_1$. As $PA_2 \perp B_1C_1D_1$ and $PB_2 \perp C_1D_1A_1$, $PA_2 \perp C_1D_1$ and $PB_2 \perp C_1D_1$. So $C_1D_1 \perp PA_2B_2$ and $C_1D_1 \perp A_2B_2$.

Inversely let the "antirespective" edges are perpendicular. Then the perpendiculars from A_2 to $B_1C_1D_1$ and from B_2 to $A_1C_1D_1$ are in the plane passing through A_2B_2 and perpendicular to C_1D_1. So they intersect. Similarly any two perpendiculars intersect and as four perpendiculars aren't in the same plane they are concurent.

c) This is an evident corollary from previous affirmation.

3. **Theorem 1.**

a) We considere the triangle ABC and two points P, Q. Let the perpendiculars from Q to AB, BC, CA intersect the lines PC, PA, PB respectively in C_1, A_1, B_1. Then he triangles ABC and $A_1B_1C_1$ are perspective-orthologic. Let A_2, B_2, C_2 are the common points of BC and B_1C_1, CA and C_1A_1, AB and A_1B_1. We denote $\overrightarrow{PA} = a$, $\overrightarrow{PB} = b$, $\overrightarrow{PC} = c$, $\overrightarrow{PQ} = x$. Then $\overrightarrow{PA_1} = \alpha a$ and $(\alpha a - x, b - c) = 0$, so $\alpha = ((x,b)-(x,c))/((a,b)-(a,c))$. Similarly $\overrightarrow{PB_1} = \beta b$, $\overrightarrow{PC_1} = \gamma c$, where $\beta = ((x,c)-(x,a))/((b,c)-(b,a))$, $\gamma = ((x,a)-(x,b))/((c,a)-(c,b))$.

As C_2 is in AB $\overrightarrow{PC_2} = \lambda a + (1-\lambda)b$. In the other way C_2 is in A_1B_1 and $\overrightarrow{PC_2} = \mu\alpha a + (1-\mu)\beta b$. By comparing the coefficients near a and b we receive that $\lambda = (1-\beta)/(\alpha-\beta)$. Now it is easy to prove that $(\overrightarrow{PC_2}, x)$ is the symmetric function of a, b, c. So for exemple $(\overrightarrow{PC_2}, x) = (\overrightarrow{PA_2}, x)$ and $A_2C_2 \perp x$.

This objection doesn't pass if P coincide with the vertex of ABC but in this case the proof is simple.

b) Let A', B', C', P', Q' are the projections of A_2, B_2, C_2, P, Q to the plane $A_1B_1C_1$. Then the triangles $A_1B_1C_1$ are perspective with center P' and orthologic with center Q'. In fact as for exemple $QC_2 \perp A_1B_1C_1$, $QC_2 \perp A_1B_1$, then by three perpendiculars theorem $Q'C' \perp A_1B_1$. By previous affirmation $P'Q'$ is perpendicular to perspective axis ℓ of triangles $A_1B_1C_1$ and $A'B'C'$ which coincide with intersection line of the planes $A_1B_1C_1$ and $A_2B_2C_2$. By three perpendiculars theorem PQ also is perpendicular to ℓ. Similarly PQ is perpendicular to all intersection lines of respective faces, so it is perpendicular to the perspective plane.

4. **Theorem 2.**

a) If the triangles are homothetic then their orthologic centers coincide with the orthocenters and the affirmation is evident. In other case it follows from previous problem.

b) The prove is similar to the previous.

2 Basic problem

5. Let A', B', C' are the touching points of the incircle with the sidelines of ABC. Then G is the perspective center of ABC and $A'B'C'$ and I is their orthologic center (two orthologic centers coincide). The affirmation follows from theorem 1.

6. Let AA_1, BB_1, CC_1 are the altitudes of ABC. Then H is the perspective center and one of orthologic centers of ABC and $A_1B_1C_1$ and O is another orthologic center.

7. Let A_2, B_2, C_2 are the touching points of the sidelines with respective excircles of ABC. The perpendiculars from A_2, B_2, C_2 to BC, CA, AB intersect in P, so the perpendiculars from A, B, C intersect to B_2C_2, C_2A_2, A_2B_2 intersect in Q and the affirmation follows from the theorem 2.

8. By the theorem 1 the trypolar of N is perpendicular to NP. Now O, M, H are collinear (Eiler line) and $HM = 2MO$. Also I, M, N are collinear (Nagel line) and $NM = 2MI$. So $IO \parallel NH$ and $NH = 2IO$. As $PI = 2OI$ $PIHN$ is the parallelogram and $IH \parallel PN$.

9. It is sufficient to note that I is the perspective center and one of the orthologic centers of ABC and $A_1B_1C_1$, and O is another orthologic center.

10. The external bissectrix intersect in the excenters I_a, I_b, I_c. I is the perspective center and one of the orthologic centers of triangles ABC and $I_aI_bI_c$, and P is their second orthologic center.

11. The affirmation follows from the fact that L is the Gergonne point of the triangle formed by the tangents to the circumcircle of ABC in the vertex.

3 Additional problems

12. Next solution was found by the participants of the conference V. Gromazin and A. Krasilnikov. Let Q is the common orthology center of the triangles $A_1B_1C_1$ and $A_2B_2C_2$, P is the common point of the lines A_1A_2 and B_1B_2, C_3 is the common point of C_1P and C_2Q. Then the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are perspective with center P. Also they are orthologic and Q is one of their orthology centers. By theorem 2 the second orthology center is in the line PQ. In other way it is in the line C_1Q, i.e. it coincides with Q. So the points C_3 and C_2 coincide and the triangles are perspective.

13. No. For exemple let A', B', C', D' are the touching points of faces with the insphere of $ABCD$. Then $ABCD$ and $A'B'C'D'$ are orthologic and the orthologic centers both coincide with the incenter of $ABCD$. But in general case the tetrahedrons aren't perspective.

14. It is easy to prove that the line PQ intersects all altitudes of tetrahedron. If the altitudes aren't collinear then such lines form the surface of degree 2 (in general case the hyperboloid) which is the sought locus. For orthocentric tetrahedron the locus is all space.

15. This exemple was found by V. Gromazin and A. Krasilnikov. Let H is the orthocenter of the triangle ABC, B_1, C_1 are the points in AB and AC such that $B_1C_1 \parallel BC$. An arbitrary line passing through H intersects AB and AC in the points M, N respectively. K is the common point of MC_1 and CH, L is the common point of BK and B_1C_1. Then $AL \perp B_1C_1$.

This result can be obtained by using of the theorem 1 to the triangles ABK and AB_1C_1.