MATHEMATICS OF TOURNAMENTS

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The problems below deal with circular tournaments, i.e. competitions of n partners matching each other (usually once). The result of any match is either win of some partner and loss of the other or draw. Some problems concern tournaments without draws. A win costs 1 point, a draw costs 1/2 and a loss costs 0. The sum of points obtained by ith paricipant in all matches will be called his result and denoted s_i.

1 Introductory problems

1. a) Prove that if $s_1 \leq \cdots \leq s_n$ then

$$\begin{aligned} s_1 + \cdots + s_n &= \tfrac{n(n-1)}{2}, \\ s_1 + \cdots + s_k &\geq \tfrac{k(k-1)}{2}, \quad k = 1, \ldots, n-1. \end{aligned} \tag{1}$$

b) Let $s_1 \leq \cdots \leq s_n$ be n nonnegative integers satisfying (1). Is it true that there exists a tournament without draws such that its participants obtain $s_1, \ldots, s_n$ points?
c) Let $s_1 \leq \cdots \leq s_n$ be n semi-integers satisfying (1).
Is it true that there exists a tournament with draws such that its participants obtain $s_1, \ldots, s_n$ points?

Definition 1. *A cycle* of length k is a set of k competitors such that 1st of them has won over 2nd, 2nd has won over 3rd,..., $(k-1)$th has won over kth, nd kth has won over 1st.

2. Prove that if a tournament without draws contains a cycle of length k then it contains at least $k-2$ cycles of length 3.

3. a) Prove that the number of cycles of length 3 in a tournament without draws equals

$$NT = \frac{n(n-1)(2n-1)}{12} - \frac{\sum_{i=1}^{n} s_i^2}{2}. \tag{2}$$

b) Determine the maximum value of NT.

Definition 2. We say that a tournament is *semi-transitive* if the following is true: if A has won over B and B has won over C then A has won over C.

4. Prove that for any tournament without draws there exists a semi-transitive tournament such that each competitor has the same results.

5. Is it true that all semi-transitive tournaments with the same s_i have the same number of:
) draws?
) non-transitive triples?

6. a) Prove that if all except one participants of a circular tournament have equal results then this competitor either has won all mathches or has lost all matches.
b) Prove that if the results are distinct for any two players and there are no draws then the participant on the first place has won all matches, on the second place — all except the match with the first one, and so on.

Definition 3. We say that a competitor is *strange* if he has won all matches with those who has better result, and has lost all matches with those who has worse result. The matches with competitors having equal result may have arbitrary outcome.

7. a) Thirty-three wrestlers have held a tournament without draws. Some strange wrestler has won all matches with those who has better result and has lost all matches with those who has worse result. No one wrestler has the same result as the strange wrestler. Prove that his place is not better than 13 and not worse than 21.
b) Prove that, in any tournament, all strange participants have the same result.

Definition 4. If some non-strange player has the same result as strange players then we call him a *middling* player. Those who has a better (resp. worse) result will be called *strong* (resp. *weak*).

c) Can a strange player take the first place? Or the last place?
d) Can the first or the last place be occupied by strange players only?

8. A circular tournament with 2^N participants contains no draws.
a) Prove that there is a chain of $N+1$ players such that the preceding player in all cases has won the match with the next one.
b) Prove that there is a chain of $N+1$ players such that any of them has won matches with all the following players.

2 Basic problems

Definition 5. We say that a triple of competitors in a tournament with draws is *non-transitive* if outcomes of matches between them form one of the following tables:

$$\begin{pmatrix} - & 1 & 0 \\ 0 & - & 1 \\ 1 & 0 & - \end{pmatrix}, \begin{pmatrix} - & 1 & 1/2 \\ 0 & - & 1 \\ 1/2 & 0 & - \end{pmatrix}, \begin{pmatrix} - & 1/2 & 1 \\ 1/2 & - & 1/2 \\ 0 & 1/2 & - \end{pmatrix}.$$

The total non-transitivity of a tournament is $NT = a + bx + cy$ where a, b, c are numbers of triples of three above mentioned kinds and x, y are some coefficients.

9. a) Determine the coefficients x, y which allow to express NT in numbers v_i, n_i, p_i of wins, draws and losses of each competitor.

   b) Find the formula for NT corresponding to such x, y.

10. a) N sportsmen have held a circular tournament. A year later the same sportsmen have held a second tournament. Prove that there is a competitor such that the change of his result does not exceed $[N/2]$.

    b) Give an example when the estimate $[N/2]$ is exact.

    c) N sportsmen, N even, have held a circular tournament twice. The change of the result of any competitor is not less than $N/2$. Prove that this change equals $N/2$ for each competitor.

11. In a tournament without draws, consider for each competitor the sum of results of those who has lost matches with him. Suppose this coefficient is the same for all players and the number of them is greater than 2. Prove that the results of all players are equal.

12. Prove that a player having the first place has the coefficient (defined in the preceding problem) not less than the average coefficient in the tournament.

13. We say that a match is *irregular* if the winner has a worse result in the tournament than the loser. What is the maximum possible part of irregular matches in an arbitrary tournament?

14. For a competitor i of a tournament, define the "Berger coefficient" B_i as the sum of results of those who has lost matches with B_i minus the sum of results of those who has won matches with B_i.

    a) Suppose $B_i \geq 0$ for all i. Is it true that $B_i = 0$ for all i?

    b) Suppose $B_i \leq 0$ for all i. Is it true that $B_i = 0$ for all i?

15. a) There were some strange competitors (see Def. 3) in a tournament. Given the results of all competitors, find the result of the strange.

    b) Prove that the number of the strange never exceeds $[N/2] - 1$.

    c) Give an example for arbitrary $N \geq 4$ and arbitrary positive integer $K \leq [N/2] - 1$ when the number of the strange is K.

    d) Prove that in any tournament the number of the strong and the strange together is less than 2/3 of the number of participants, and the same is true for the number of the weak and the strange together.

    If, in addition, there are no middling competitors then the number of the strong and of the weak separately is greater than 1/3, and the number of the strange is less than 1/3 of the number of participants. More precisely, if N is of the form $3M$, $3M + 1$, or $3M + 2$ (M positive integer) then the number of the strange does not exceed $M - 2$, $M - 1$, M respectively.

e) Give an example (for arbitrary $N \geq 5$) when there are some strange, no middling, and the number of the strong as well as of the weak equals the least integer exceeding $N/3$.

f) Suppose any two participants of a circular tournament have distinct results. Give an example when there are exactly two draws and a strange competitor.

g) Suppose a circular tournament contains exactly one draw, and any two competitors have distinct results. Prove that there are no strange competitors.

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3 Additional problems

16. For a tournament with given s_i, determine:

    a) the maximum possible number of draws;

    b) the minimum possible number of non-transitive triples.

17. We say that a tournament is *regular* if the following is true: if A has won the match with B then A has won all the matches with players whose result is not better than that of B, and B has lost all the matches with players whose result is not worse than that of A.

    a) Prove that any regular tournament is semi-transitive.

    b) Prove that a semi-transitive tournament is not regular iff there are four players with the table of results of mutual matches of one of the following forms:

$$\begin{pmatrix} - & 1/2 & 1 & 1 \\ 1/2 & - & 1/2 & 1/2 \\ 0 & 1/2 & - & 1 \\ 0 & 1/2 & 0 & - \end{pmatrix}, \begin{pmatrix} - & 1/2 & 1/2 & 1 \\ 1/2 & - & 1 & 1/2 \\ 1/2 & 0 & - & 1/2 \\ 0 & 1/2 & 1/2 & - \end{pmatrix}.$$

    c) Give an example of the distribution of results not possible in any regular tournament.

    d) Give the conditions on the distribution of results to correspond to some regular tournament.

18. For a tournament without draws [and with more than two participants], consider the coefficients of players as defined in Problem 11. Further, consider the second coefficient which equals the sum of the coefficients of players who has lost matches with the given player. Also consider the third coefficient equal to the sum of the second coefficients of these players, and so on.

    a) Suppose kth coefficients of all players are equal for some k. Is it true that all competitors have equal result?

    b) Consider orderings of the players corresponding to their first, second and so on coefficients. Prove that after some moment the ordering will not change.

19. We'll call A a common winner for $B_1, B_2, \ldots, B_k$, if he won parties with all this players. We'll say, that a table of tournament satisfies $P_{n,k}$ condition, if every k players have a common winner.

    a) Every 2 participants have a common winner. Prove that $n \geq 7$.

    b) In a tournament with 7 participants 21 match is played, so there exists 3^{21} possible tables of tournament. How much of these tables satisfy the $P_{7,2}$ condition?

    c) Give an example of a tournament, which satisfies $P_{n,3}$ condition (for some n).

    d) Prove that there exists a table of tournament that satisfies the $P_{n,5}$ condition (for some n).

4 Introductory problems

1. a) Any k competitors hold $\frac{k(k-1)}{2}$ matches between them and in each match the sum of points of both partners is 1. Hence the sum of points of any k competitors is not less than $\frac{k(k-1)}{2}$. For $k = n$ this inequality clearly turns to equality.

   b) Given an admissible set $s_1, \dots, s_n$, consider all differences of the form $s_1 + \cdots + s_k - \frac{k(k-1)}{2}, k = 1, \dots, n-1$. Order admissible sets according to the maximum difference and in the case of equality according to the number of maximum differences in the set. Now we use induction using obtained ordering.

   If $s_1 + \cdots + s_k = \frac{k(k-1)}{2}$ for all k then $s_1 = 0, s_2 = 1, \dots, s_n = n-1$ and the construction of the tournament is obvious. Let $s_1 + \cdots + s_k\frac{k(k-1)}{2} = l > 0$ be the maximum difference. Then $s_1, \dots, s_k - 1, s_{k+1} + 1, \dots, s_n$ is an admissible set preceding to the given one. By induction, there exists a tournament with such set of results. If in this tournament $(k+1)$th competitor has won over kth then replace the outcome of their match by the opposite one. Now suppose kth player has won. Since $s_{k+1} + 1 > s_k - 1$, some competitor has won over kth but has lost the match with the $k+1$th. To obtain the desired tournament, replace the outcome of these matches by the opposite one.

   c) As in b), use induction on the number m of semi-integer s_i's. Note that this proves a stronger assertion: there exists a tournament with given set of results including exactly $m/2$ draws.

2. Use induction on k. The case $k = 3$ is trivial. Suppose $k > 3$ and the assertion is true for $3, \dots,$ $k-1$. Consider matches between 1 and 3, 2 and 4,..., $k-2$ and k, $k-1$ and 1, k and 2. If all these matches are won by the partners named second, then k triples $(1,2,3), (2,3,4), \dots,$ $(k-1,k,1), (k,1,2)$ are not transitive, and our assertion is true. Suppose kth player has won over 2nd. Then the players $2, 3, \dots, k$ form a cycle of length $k-1$, and by induction there exist $k-3$ non-transitive triples consisting of these players. Now consider the triples $(1,2,3), (1,3,4), \dots,$ $(1, k-1, k)$. We will show that at least one of them is not transitive. Suppose the contrary. Since 1 has won over 2 and 2 over 3, 1 has to win over 3; since 3 has won over 4, 1 has to win over 4 and so on. Finally, 1 has to win over k but this contradicts the statement of the problem. Hence at least one of the mentioned triples is not transitive. Since it coincides with one of $k-3$ triples mentioned before, the assertion is proved.

3. a) Suppose all matches have already been played except the match between 1 and 2. Clearly, the result of the last match concerns transitivity of those triples only in which both 1 and 2 are involved. Suppose these players have obtained t_1 and t_2 points respectively in matches with the rest of competitors. Further, suppose they both have won over x_{11} players; 1 has won and 2 has lost the matches with x_{12} players; 2 has won and 1 has lost the matches with x_{21} players; and both have lost matches with x_{22} players. Clearly, the following is true:

   $$x_{11} + x_{12} = t_1, \quad x_{11} + x_{21} = t_2, \quad x_{11} + x_{12} + x_{21} + x_{22} = n-2.$$

   So if we know t_1, t_2 and one of the numbers $x_{11}, x_{12}, x_{21}, x_{22}$ then we can determine all the others. In particular, $x_{12} = n - 2 - t_2 - x_{22}$, $x_{21} = n - 2 - t_1 - x_{22}$.

   Now consider possible outcomes of the last match.

   1) The victory of 1. Then 1 obtains $t_1 + 1$ points, 2 obtains t_2, and the right side of (2) [see the statement of the problem] takes the form

   $$C_1 - (t_1+1)^2/2 - t_2^2/2$$

   where C_1 is half-sum of squared results of the rest of competitors and does not depend on the outcome of the last match.

A triple including 1 and 2 is transitive iff its third participant wins over 1 and loses the match with 2. The number of such triples is x_{21}, hence the left side of (2) equals $C_2 + x_{21}$ where C_2 does not depend on the outcome of the last match.

2) The victory of 2. Similarly, we obtain for the right and for the left side of (2) respectively:

$$C_1 - (t_2+1)^2/2 - t_1^2/2,$$

$$C_2 + x_{12}.$$

Making use of expressions for x_{12}, x_{21} obtained above, we see that the difference between the left and the right side in both cases equals

$$C_2 - C_1 + t_1^2/2 + t_2^2/2 + 1/2 + n - 2 - x_{22}.$$

Hence this difference does not depend on the result of an arbitrary match and so is equal for all tournaments. Now consider the tournament in which the first player has won over all others, the second over each except the first one and so on. Then the number of non-transitive triples is 0, and the results are $s_1 = n-1, s_2 = n-2, \dots, s_n = 0$. Making use of the formula for the sum of squares of positive integers, we see that (2) holds for this tournament. Hence it holds for all tournaments.

b) Replace the outcomes of all matches by the opposite ones. The transitivity of triples will not change, and the results of the competitors will be $n-1-s_1, \dots, n-1-s_n$. Hence

$$\begin{array}{rcl} NT & = & \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum s_i^2 = \frac{n(n-1)(2n-1)}{12} - \frac{1}{2}\sum (n-1-s_i)^2 = \\ & = & \frac{n(n-1)(2n-1)}{12} - \frac{1}{4}\sum (s_i^2 + (n-1-s_i)^2) \end{array}.$$

Since $s_i^2 + (n-1-s_i)^2 = \frac{(n-1)^2+(2s_i-n+1)^2}{2}$,

$$NT = \frac{n^3 - n - 3\sum(2s_i - n + 1)^2}{24}.$$

Thus for n odd $NT_{max} = \frac{n^3-n}{24}$ which corresponds to $s_i = (n-1)/2$ (such a tournament is easily constructed inductively). For n even, $2s_i - n + 1$ is odd, hence its square ≥ 1. The value $NT_{max} = \frac{n^3-4n}{24}$ is achieved in a tournament where a half of competitors has $s_i = n/2$, and the other half has $s_i = (n-2)/2$. Such a tournament is obtained by adding a single player to a tournament with odd number of participants all having equal results.

4. Consider the tournament with given distribution of results which has the maximum number of draws. Suppose A has won over B and B over C. If C has won over A then replace the results of the three matches by draws. If A and C have drawn then replace the outcomes for $A-B$ and $B-C$ by draws, and the outcome for $A-C$ by the victory of A. In both cases the results of the rest of players do not change and the number of draws increases which contradicts the maximality of the tournament.

5. No. For exemple consider two tournaments:

$$\begin{pmatrix} - & 1/2 & 1 & 1/2 & 1 & 1/2 \\ 1/2 & - & 1/2 & 1 & 1/2 & 1 \\ 0 & 1/2 & - & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/2 & - & 1/2 & 1 \\ 0 & 1/2 & 0 & 1/2 & - & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 1/2 & - \end{pmatrix}, \begin{pmatrix} - & 1/2 & 1/2 & 1/2 & 1 & 1 \\ 1/2 & - & 1/2 & 1/2 & 1 & 1 \\ 1/2 & 1/2 & - & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & - & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & - & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 1/2 & - \end{pmatrix}$$

6. a) Suppose the "non-standard" competitor has better result than "standard" ones. The average result is $(N-1)/2$, and its difference with the "standard" result is $\geq 1/2$, so the "standard" players have total shortage not less than $(N-1)/2$. Te "non-standard" player exceeds the average level by the same value, hence he has $\geq N-1$ points. This is possible in the only case when the "non-standard" player has won over all the "standard".

The case when the "non-standard" player has worse result than "standard" ones is treated similarly.

b) If there are no draws then all results are integer and belong to the interval $[0; N-1]$. If they are pairwise distinct then they include all integers of this interval. Clearly, the player with $N-1$ points has won all matches. Then the player with $N-2$ points has won over everybody except the first player, and so on.

7. a) Let the strange have a place better than 13. Then the number of those who has a worse place is greater than 20. Consider matches between them. Each of them participated in ≥ 20 such matches. At least one of of these players has won in a half or more of these matches, thus obtaining ≥ 10 points. Also he has won over the strange, and hence his result is ≥ 11. The strange has a better result, that is ≥ 12 points. But the strange may won over those only who has a better result. So they are ≥ 12 in number, and the place of the strange is not better than 13.

   Now replace the outcomes of all matches by the opposite ones. The assumptions of the problem are fulfilled as well, so by the proved above the strange has the place not better than 13. Again replace the outcomes by the opposite ones. The order of results turns to the inverse. It remains to note than 21th place among 33 competitors is the 13th from the end.

   b) Suppose A and B are strange players, and A has a better result than B. Then B has won over A. Clearly A has better outcome than B in a match with some (and not only one) player C. But if C has worse result than A than C has won over A (by definition of the strange). And if C has not worse result than A then C has better result than B. Hence B (as strange) has won over C. Thus A has not better outcome than B in a match with any player, in contradiction with the proved above.

   c) Consider a tournament with 6 participants. Three of them have mutually drawn as well as the fourth and the fifth which both have lost their matches with the three first. The last player has won over the three first and has lost his match with the fourth and the fifth. Then the sixth shares the first place with the three first and is strange.

   To place the strange at the last place, replace all the outcomes by the opposite ones.

   d) Suppose strange players occupy the first place entirely and are M in number. Any of them has lost his matches with all players not sharing the first place and thus has result not better than $M-1$. The rest of players have won over all the strange and so have results not worse than M but this contradicts the mentioned just above. Hence the first place cannot be occupied by the strange only.

   The argument for the last place is similar.

8. a) Use induction on N. The start ($N=1$) is trivial. To proceed from $N-1$ to N, divide 2^N competitors into disjoint pairs. Consider the winners in these pairs and the matches between them. By induction, there exists a chain of N such competitors in which any member has won over the next one. The last member is the winner of some pair considered above. Adding the loser of this pair, we obtain the desired chain of length $N+1$.

   b)Use induction on N. The start ($N=1$) is trivial. We shall proceed from $N-1$ to N. Suppose a sportsman S is the(or a) winner in a tournament with 2^N participants and has won over $M0$ players. This number is $\geq 2^{N-1}$ competitors (otherwise any participant of the tournament has won less matches than lost but the total number of wins has to be equal that of losses). Choose some 2^{N-1} players of $M0$ and denote this set by $M1$. By induction, $M1$ contains a chain of length N with the desired property. Adding the player S to the beginning of this chain we obtain the desired chain of length $N+1$.

5 Basic problems

9. a) Consider two pairs of tournaments.

$$\begin{pmatrix} - & 1/2 & 1/2 & 1/2 & 1 \\ 1/2 & - & 1 & 0 & 1/2 \\ 1/2 & 0 & - & 1 & 1/2 \\ 1/2 & 1 & 0 & - & 1/2 \\ 0 & 1/2 & 1/2 & 1/2 & - \end{pmatrix}, \begin{pmatrix} - & 1 & 1/2 & 1/2 & 1/2 \\ 0 & - & 1 & 1/2 & 1/2 \\ 1/2 & 0 & - & 1 & 1/2 \\ 1/2 & 1/2 & 0 & - & 1 \\ 1/2 & 1/2 & 1/2 & 0 & - \end{pmatrix}$$

and

$$\begin{pmatrix} - & 1/2 & 1/2 & 1 & 1 \\ 1/2 & - & 1 & 0 & 1 \\ 1/2 & 0 & - & 1 & 1/2 \\ 0 & 1 & 0 & - & 1/2 \\ 0 & 0 & 1/2 & 1/2 & - \end{pmatrix}, \begin{pmatrix} - & 1 & 1 & 1/2 & 1/2 \\ 0 & - & 1/2 & 1 & 1 \\ 0 & 1/2 & - & 1 & 1/2 \\ 1/2 & 0 & 0 & - & 1 \\ 1/2 & 0 & 1/2 & 0 & - \end{pmatrix}.$$

In both pairs, v_i, n_i, p_i coincide for all i. Examine all the triples and equate summary non-transitivities for each pair to obtain the following:

$$\begin{cases} 1 + 9y &= 3x + 6y \\ 1 + 2x + 3y &= 4x + 3y \end{cases}$$

The solution of this system is $x = 1/2$, $y = 1/6$.

b) Arguing as in Problem 3 we obtain:

$$NT = \frac{((n^3 - n) - \sum n_i(n_i + 2) - 3\sum(v_i - p_i)^2)}{24}$$

10. a) For brevity, set $M = [N/2]$. Suppose there are X competitors in a tournament having increased their result by more than M. Denote the sum of these increases by D. Then $D > XM$. The sum of outcomes of matches between given players is always the same, and the increase is obtained in matches with other players only. It achieves its maximum if X competitors above have lost all their matches with other players in the first tournament and have won all of them in the second. In this case $D = X(N - X)$. Thus $XM \leq X(N - X)$. If $X > 0$ then $X < N - X$. For $X = 0$ this is also true. Since X is integer, this implies $X < N/2$.

Now let Y be the number of participants having decreased their result by more than M each. Arguing as above we see that $Y < N/2$. Summing up two last inequalities, we obtain the desired result.

b) Let the 1st player win over all others, the 2nd over all except the 1st and so on (ith player has obtained $N - i$ points). Suppose the table of the second tournament is similar but all the players are shifted circularly by M (here $M = [N/2]$ as above): the 1st has passed to the $(M + 1)$th place, the 2nd to the $(M + 2)$th, ..., the $(N - M)$th to the last, the $(N - M + 1)$th to the 1st, ..., the Nth to the Mth place. Then those who had the best $N - M$ results in the first tournament have decreased them by M, and all others have decreased them by $N - M$. But $N - M \leq M$, so the minimum change is M.

c) The participants of the tournament divide into two groups, namely those who has increased (resp. decreased) their result by $N/2$. Some of these groups has to include $\geq N/2$ sportsmen. Let it be the first group, and let it contain X sportsmen. If the summary increase of their results is D then $D \geq XN/2$. This increase is entirely due to matches with the other $N - X$ competitors, hence $D \leq X(N - X)$. This implies $N - X \geq N/2$. By assumption $X \geq N/2$, thus $X = N/2$ and both inequalities turn to equalities. First of them means that each member of the first group has increased his sum by $N/2$ exactly. The proved above implies that the second group consists also of $N - X = N/2$ members, so we may argue similarly.

11. Suppose not all results are equal, the first place corresponds to K and the last to L points. The coefficient of the competitors occupying the 1st place is the sum of K numbers each not less than L. Hence this coefficient is $\geq KL$. Similarly, the coefficient of the competitors occupying the last place is the sum of L numbers each not greater than K, hence it is $\leq KL$. Being equal, these two coefficients are equal to KL. This implies that the players at the 1st (resp. last) place have won only over (some) players having L (resp. K) points, that is occupying the last (resp. 1st) place.

    If the 1st place is shared by more than two or more sportsmen then one of them has won over the other, in contradiction with the above. Thus the 1st place is occupied by a single player, and similarly the last place. By assumption, there is some third competitor in the tournament. By the above, he has won over both the 1st and the last player. But then he has better result than the winner since the winner could win over the last player only. This provides the desired contradiction.

12. Let N be the number of participants of a tournament minus 1 (i.e. the number of matches of each participant). One of players sharing the 1st place will be called "the champion". Those who have loss (resp. win) in the match with him will be simply called "losers" (resp. "winners"). Set $M = [(N^2-1)/4]$. We will show that the champion's coefficient is $\geq M$. Suppose his result is K. Then the number of "winners" is $N-K$, their results are not better than that of the champion, so the sum of the results of "winners" does not exceed $(N-K)K$. The total sum of results is $N(N+1)/2$, hence the sum of the results of "losers" is $\geq N*(N+1)/2 - K - (N-K)*K = N*(N+1)/2 - K*(N+1-K) \geq N*(N+1)/2 - (N+1)^2/4 = (N+1)/2*(N-1)/2 = (N^2-1)/4$. Since the champion's coefficient is integer, it is not less than M.

    Now we proceed to show that the average coefficient of the competitors does not exceed M thus proving the desired assertion. If a player has obtained X points then he increases by X the coefficients of those who has won over him, and the total sum of coefficients is increased by $X(N-X)$. This number does not exceed $N^2/4$. Being integer, for odd N it does not exceed $(N^2-1)/4$. Thus it does not exceed M.

    The total sum of coefficients is obtained by summing $(N+1)$ such numbers. To obtain the average coefficient, we have to divide this sum by $N+1$, thus obtaining a number $\leq M$ as required.

13. First we prove that the part of irregular matches is always less than $\frac{3}{4}$. Let n be the number of players, $m = [n/2]$. We call *the strong* (resp. *the weak*) those who occupies the first m places (resp. all other places). (The players with equal results are ordered arbitrarily.) Denote by x the number of regular matches between the strong and the weak. The sum of points obtained by the strong in the matches with the strong equals $m(m-1)/2$ and in the rest of matches does not exceed x. So the average result of a strong player does not exceed $(m-1)/2 + x/m$. Similarly, the average result of a weak player is not less than $(n-m-1)/2 + (m(n-m)-x)/(n-m) = (n+m-1)/2 - x/(n-m)$. If irregular matches are present then not all results are equal, hence the average result of the strong is greater than of the weak. Thus $x > m(n-m)/2 \geq (n^2-1)/8$. Since the total number of matches equals $n(n-1)/2$, the part of regular matches exceeds $(n+1)/4n > 1/4$.

    Now we proceed to prove that the part of irregular matches can be arbitrarily close to $\frac{3}{4}$. Consider the following tournament of $2k+1$ competitors: a player with arbitrary number $i \leq k$ loses his matches against players with numbers $i+1, \ldots, i+k$ and wins over all others, and each player with number $i > k$ wins over players with numbers $i-k, \ldots, i-1$ and loses all other matches. Now "multiply" each player replacing him by a couple of k new players, and let the players of distinct blocks have the same mutual outcomes as the corresponding former players and all the matches inside a block are draws. Let the players of each block be numbered by $1, \ldots, k$. Change results of matches between blocks $k+1$ and $k+1-i$ $(i = 1, \ldots, k)$ in the following way. The player number j of the block $k+1$ for $j \leq k-i$ draws matches with the players $j+1, \ldots, j+i$ and wins over the rest of the block $k+1-i$, and for $j > k-i$ he wins over the players $j-k+i, \ldots, j-1$ and draws with the rest. In the matches between blocks $k+1$ and $k+1+i$ replace ik losses by draws in a similar way. Now the sum of results is the same inside each block and decreases when the number of the block increases. The number of irregular matches equals $\frac{3}{2}k^4 - \frac{k^3}{2} - k^2$. Since the total number of matches is $k(2k+1)(k(2k+1)-1)/2$, for great k the part of irregular matches tends to $\frac{3}{4}$.

14. Consider the sum $\sum_i s_i B_i$. The definition of B_i implies that this sum consists of summands of the form $\pm s_i s_j$ according to all pairs of not-drawing partners. Each of these products is contained twice with opposite signs, so the sum is 0. If all B_i have the same sign, then they are not zero only if $s_i = 0$. But $s_i = 0$ implies $B_i < 0$, so the question (a) has positive answer. The question (b) has negative integer. Suppose for example that some player has lost all his matches, and all other matches are drawn. Then the coefficient of the outsider is negative, and all others are 0.

15. a) A strange player obtains his points in matches with those only who has not worse result. Thus he has the following property:

    (*) *The result of a strange player is less than the number of players (including the given one) having the same or better result.*

    Suppose a player A has better result than a strange player (who by definition wins over all competitors having better results). Then the result of A exceeds the number of those who has not worse result. Thus A does not have the property (*).

    The answer is now clear: the result of a strange player is the maximum result which is less than the number of competitors having the same or better result.

    Note that we have proved once more that all the strange have equal results.

    b) All strange competitors share the same place in the tournament table. Suppose this is not the 1st place. Then the player at the 1st place has lost all his matches with the strange. But he has lost less matches than won. So the number of his losses is less than $N/2 - 1$ and (as an integer) not greater than $[N/2] - 1$ as required. If strange players share the 1st place then we may apply the similar argument to the last place.

    c) Suppose $M = [N/2]$, $1 \leq L \leq M - 1$, the competitors are numbered from 1 to N and are divided into 4 groups including respectively the participants with numbers from 1 to L, from $L+1$ to M, from $M + 1$ to $N - L$, and the rest. Further, suppose the first group has lost all matches with the second group and has won all matches with the third and the fourth group. The fourth group has won all matches with the second and has lost all matches with the first and the third. All other matches are drawn. It is easy to check that each player of the first group has obtained $N - M + (L-1)/2$ points, that is $\geq N/2$. Each player of the fourth group has obtained $M - (L+1)/2$ points, that is $\leq N/2 - 1$. Each player of the second and of the third group has obtained $(N - 1)/2$ points. Thus the second group consists of the strange, the first group of the strong, the third group of the middling and the fourth group of the weak. The number of the strange players equals $[N/2] - L$, and an appropriate choice of L allows to make it any integer from 1 to $[N/2] - 1$.

    d) Consider the "average" strong competitor (i.e. sum the results of the strong and divide by their number). This imaginary player has equal number of wins and losses in the matches with the strong. Furthermore he has lost all matches with the strange. Thus his result does not exceed the number of the middling and the weak plus the half of the number of the strong. Similarly, the result of the "average" strange competitor is not less than the number of the strong plus the half of the number of the strange. But this sum is less than the first one. Thus the number of the middling and the weak together exceeds the half of the number of the strong and the strange together. This means that the strong and the strange consist less than 2/3 of the total number of the competitors. The assertion concerning the number of the strange and the weak together is proved similarly.

    e) In fact, the construction of an example follows from the solution of (d). Divide the competitors into three groups, each of two first including $[N/3] + 1$ players. Suppose the players of the 1st group have won all matches with the players of the 2nd group. Similarly, the 2nd group has won over the 3rd and the 3rd over the 1st. All matches inside groups are draws. It is easy to check that the 3rd group consists of the strange, the 1st of the strong and the 2nd of the weak.

    f) Let the number of competitors be odd, they are numbered from 1 to $2M + 1$, and the lesser numbers have won over the greater with the following exceptions. Players number 1 and M have drawn as well as $M + 2$ and the last. The player number $M + 1$ has won over all the lesser numbers and has lost all matches with the greater numbers. It is easy to check that the result decreases as the number of the competitor increases and that the number $M + 1$ is strange.

g) We need the following

Lemma. *Suppose all the participants of a circular tournament have distinct results and [totally] have drawn exactly one match. Then*

1) a difference between two neighboring results does not exceed 2;

2) if a difference between two neighboring results equals 2 then these players have not made a draw.

Proof of the lemma. Suppose the assertion is false. Choose a tournament with minimum number of participants not satisfying the lemma. Obviously $N > 2$. For brevity, a difference between two neighboring results will be called simply "a difference".

Let S be the sum of all differences, D a difference contradicting 1) or 2). Denote the number of semi-integer differences between other players by K. Each semi-integer (resp. integer) difference is not less than 1/2 (resp. 1). Thus

$$(**) \qquad S \geq D + N - 2 - K/2\ .$$

The results of the players involved (resp. not involved) in the draw are semi-integer (resp. integer). Hence each semi-integer difference involves some of the "participants" of the draw, and the number of these differences does not exceed 4.

Suppose 1) fails, i.e. $D > 2$. If $D \geq 3$ then in view of (**) $S \geq N - 1$ (because $K \leq 4$). If $D = 5/2$ then $K \leq 3$ (because D is semi-integer), so $S \geq N - 1$ again. Now suppose 2) fails, i.e. $D = 2$ and some "participant" of this difference is involved in the draw. His result is semi-integer, hence the second "participant" of the difference D also has a semi-integer result and thus is involved in the draw. Consequently, $K \leq 2$ and in view of (**) $S \geq N - 1$ again.

Thus in all cases the difference between the best and the worst result is not less than $N - 1$. But it is never greater, and the equality means that some competitor has won all his matches and some competitor has lost all them. The difference D does not involve at least one of the "extreme" competitors. Remove him from the tournament. The assertion of the lemma still fails but the number of participants has diminished. This contradiction proves the lemma.

Now suppose the results of all participants are distinct, there is exactly one draw and some player A is strange. According to Problem 2b he does not have the 1st or the last place. The results of the neighboring players B and C differ from the result of A by not less than 1/2. So the difference between them is not less than 1.

Remove A from the tournament. The results of the weak (who have won over A) decrease by 1, and the results of the strong do not change. Thus the results of all the participants are still distinct. The draw is still present since A (as strange) is not involved in it. The results of B and C are now neighboring, and their difference is not less than 2. By the lemma, it equals 2 and B and C are not involved in the draw.

Hence the initial results of B and C differ by 1 and so differ by 1/2 from the result of A. But A, B and C are not involved in the draw and thus their results are integer. This contradiction proves the assertion of the problem.

16. It is an unsolved problem.

17. a) Hint: if A beats B then one may prove that A got more scores than B.

    c) Consider, for example, a tournament, where 5 participants got respectively 3, 3, 2, 1.5 and 0.5 scores.

18. a) Answer: yes.

19. a) Hint: if minimal quantity for finding common winner for every r participants, equals n, then for $r + 1$ one needs at least $2n + 1$ participants.

    b) Answer: 240.

c) Solution of p. a) shows, that one need at least 15 participants. One may prove, nevertheless, that 15 is not enough. We know the example for $n = 19$. The method: let us numerate participants by $0, 1, \ldots, 18$ and decide, that a beats b, if $a - b \equiv r^2 mod 19$.

Unsolved problem: if you'll take another $p = 4k + 3$ instead of $p = 19$, then the same construction gives us table, in which every r participants have a common winner, if $p = p(r)$ is sufficiently great.

d) One may prove the existence of a table with such a condition in such a way: let N (number of participants) is sufficiently great (one may take, for examle, $N = 2000$) and consider ALL the tables for N participants. Let us find the quota of tables in which given 5 participants don't have a common winner. It equals $(1 - \frac{1}{32})^{N-5}$. So it's very small, and one easily proves that the quota of tables, in which every five have a common winner, is more than zero. Krasnodar students proposed another, more effective method of building the table, but their method requires an enormous quantity of participants.