REALIZATION OF GRAPHS WITH ROTATIONS

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Introduction and motivation.

The problem of embeddability (i.e. realizability without self-intersections) of graphs (or graphs with an additional structure) into the plane, the torus and the Möbius band (and other surfaces) is one of the most important in the topological graph theory. A nice form of a solution of this problem is to find a number of graphs such that a graph G would be realizable if and only if G does not contain any of these graphs. In this sequence of problems we present some results of this kind. The main results are problems 2e and 8ab (folklore, [ReSk00]), X5b [Man05, Per05], 9c [Per]. Problem 4c is unsolved.

In the following problems, let us agree that for any assertion formulated as a problem, a proof of this assertion is required.

Introductory problems. The planarity of graphs.

In this section we recall basic notions of graph theory. The reader familiar with graph theory can omit this section and return here upon necessity. It is allowed to use without proof the Jordan Theorem, the Euler Theorem and its generalizations; these results are stated in the text.

A *graph* is a finite set in which certain subsets with multiplicities are chosen; these subsets should consist of one or two elements. The elements of this set are called *vertices* of the graph. The chosen subsets are called *edges* of the graph.

The vertices are depicted as points (e.g. in the plane or in the space). Each edge corresponding to a chosen subset consisting of 2 elements is depicted by a broken line (or a curve) joining the corresponding points. Each edge corresponding to the chosen set $\{A\}$ is depicted by a closed broken line (or a curve) joining the vertex A to itself. Thus we admit loops and multiple edges. On a picture broken lines could possibly intersect, but the intersection points (that are not ends of an edge) are not vertices. For simplicity we shall only consider those pictures on which the edges are depicted by broken lines (but not arbitrary curves).

A *path* in a graph is a finite sequence of vertices in which any two subsequent vertices are joined by an edge. A *cycle* is a path in which the first and the last vertices coincide. A graph is *connected* if each two of its vertices could be joined by a path. A graph is a *tree* if it is connected and does not contain cycles without self-intersections. Clearly, each connected graph contains a tree containing all vertices of the graph. Such a tree is called *maximal tree*.

A graph is called *planar* if it can be drawn in the plane so that the interiors of the edges (i.e. edges without their endpoints) are disjoint and do not have self-intersections. E.g. any tree is planar and any graph formed by vertices and edges of a convex polyhedron is planar. The graphs K_5 and $K_{3,3}$ (fig. k5k33) are not planar. This can be proved using the following Jordan Theorem.

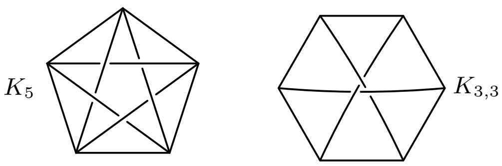

Figure k5k33

The Jordan Theorem. *A closed curve without self-intersections in the plane splits the plane exactly in two parts. (One of these parts is bounded and the other unbounded. Two points of the plane not lying on the curve are contained in the same part if and only if they could be joined by a broken line disjoint with the curve).*

A *plane graph* is a specific mapping (i.e. a picture or a realization) of the graph in the plane without self-intersections. Sometimes this mapping is called just a *graph*, but this is unfair because one and the same planar graph can be drawn (without self-intersections) in the plane in several ways (fig. dif). The Jordan Theorem implies that *any plane graph splits the plane into a finite number of connected components.* These components are called *faces* of the plane graph.

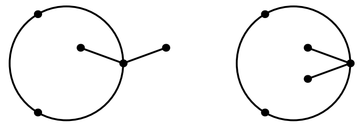

Figure dif

The Euler formula. *For a connected plane graph having V vertices, E edges and F faces we have $V - E + F = 2$.*

0. Applications of the Euler formula. (a) K_5 and $K_{3,3}$ are not planar.

(b) In any planar graph distinct from a point $2E \geq 3F$ and $E \leq 3V - 6$.

(c) In any planar graph there is a vertex of degree ≤ 5.

(d) In any plane graph there is either a triangular face or a vertex of degree 3.

Informally, a *subgraph* of a given graph is a part of this graph. Formally, a graph G is a *subgraph* of the graph H if each vertex of G is a vertex of H and also each edge of G is an edge of H.

Clearly, any subgraph of a planar graph is planar. Clearly, there exists a non-planar graph which does not contain a subgraph, isomorphic to K_5 or $K_{3,3}$.

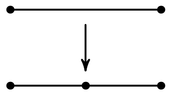

Figure podr

The *subdivision of an edge* operation is shown in fig. podr. Two graphs are called *homeomorphic*, if one can be obtained from the other by means of subdivisions of edges and inverse operations.

Clearly, two homeomorphic graphs are either both planar or both non-planar.

The Kuratowski Theorem. *A graph is planar if and only if it does not contain any subgraph homeomorphic to K_5 or $K_{3,3}$ (fig. k5k33).*

Introductory problems. Realization of graphs with rotations.

A *half-edge* of the graph is a pair of an edge and its endpoint (or, informally, an end of an edge). A *graph with orientable rotations* (shortly: *an R-graph*) is a graph for each of whose vertices a cyclic ordering of the edges issuing from this vertex is fixed. (Synonyms: a graph with the system of rotation, a sketch.)

1. (a) How many cyclic orderings are there on the set of four numbered vertices of K_4 (i. e. how many R-graphs correspond to the ordinary graph K_4 with numbered vertices)?

(b)* The same question for non numbered vertices (i. e. how many R-graps up to R-isomorphism correspond to the ordinary graph K_4).

An R-graph is called *R-planar*, if it can be drawn without self-intersections in the plane so that for any vertex the cyclic ordering of edges in the clockwise direction around this vertex coincides with the fixed cyclic ordering. For R-graphs the notions of R-subgraph, being R-homeomorphic etc. are defined analogously to ordinary graphs.

2. (a) Which R-graphs corresponding to the graph K_4 are R-planar?

(b) Any tree with any rotations is R-planar.

(c) Any R-tree with n vertices can be drawn respecting rotations and without self-intersections outside open polygon $A_1 \dots A_n$ in the plane so that the vertices of the tree would be vertices of the polygon.

(d) Given an R-graph, take its maximal R-tree. Colour the edges outside the maximal tree in red. From each red edge we shall cut his 'middle third' as in fig. tret. Then we obtain an R-tree T. Draw it in the plane as in (c). Suppose that there are vertices A, B, A_1, B_1 of T lying on the boundary of the polygon in this order so that vertices A and A_1 are in one red edge, while B and B_1 are in another red edge. Then the initial R-graph contains R-subgraph, R-homeomorphic to one of those shown in fig. 8theta.

(e) **The R-planarity Criterion.** An R-graph is R-planar if and only if he does not contain R-subgraph, R-homeomorphic to one of the represented on fig. 8theta.

Remark: the necessity is already proved (using Jordan theorem) during presentation.

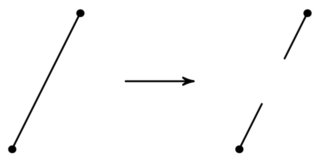

Figure tret

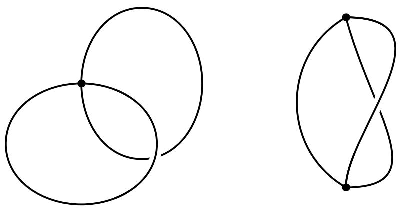

Figure 8theta

3. The torus, the Möbius strip, the sphere and the cylinder (i.e. the annulus) are drawn in fig. tor.

(a) Is it possible to draw without self-intersections (i.e. to embed or to realize) graph K_5 in the torus?

(b),(c) The same question for the sphere and the cylinder (i.e. the annulus).

(d) Is it possible to draw without self-intersections graph $K_{3,3}$ on the Möbius strip?

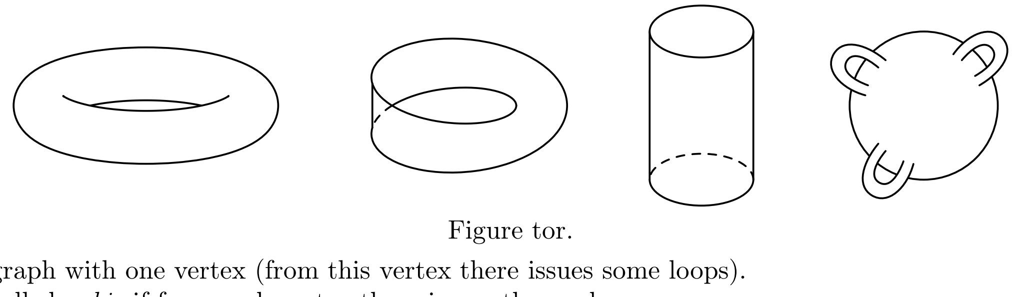

Figure tor.

A *wedge* is graph with one vertex (from this vertex there issues some loops).

A graph is called *cubic* if from each vertex there issues three edges.

4. (a) An R-wedge is R-planar if and only if it does not contain the 'eight' figure (fig. 8theta).

(b)* Find an analogue of criterion (a) for R-embeddings (R-realizations) of R-wedges in the torus.

(c)* Find an analogue of the R-planarity criterion for R-embeddings of cubic R-graph in to the torus.

Introductory problems. Realization of graphs with cross-structure.

The *degree* of a vertex is the number of edges going issuing out of this vertex.

An *Euler cycle* is a cycle passing through each edge exactly once.

Suppose that each vertex of a graph G has degree 4. Let T be an Euler cycle in this graph. Take a new graph $S(T)$, which is a cycle without self-intersections, whose edges correspond to all the edges of T and go in the same order. In this way to each vertex of G there correspond two vertices of $S(T)$. For each vertex of G join by an edge the pair of corresponding vertices of $S(T)$. Such edges are called *chords*. The union of the graph $S(T)$ and all the constructed chords is called a *chord diagram*. Two chords are called *skew* if the corresponding vertices P and Q of G go along T in the order $PQPQ$.

X1. (a) A chord diagram (of an Euler cycle in a graph of degree 4) is planar if and only if the set of its chords can be broken into two subsets so that any two chords from one set are not skew.

(b) A graph of degree 4 is planar if and only if when in it has an Euler cycle whose chord diagram is planar.

A *half-edge* of the graph is a pair of an edge and its endpoint (or, informally, an end of an edge). A *graph with cross-structure* (shortly: an X*-graph*) is a graph whose vertices have degrees 4, and for each of them a splitting of the correponding four edges into two pairs is fixed. In other words, a cross structure on a graph of degree 4 is a colouring of each half-edge into black or white such that out of each vertex there issues two black and two white half-edges.

X2. In any X-graph there exists an Euler cycle such that along this cycle at each vertex the colors of half-edges change.

Suppose that we have two paths without common edges. A common vertex of these paths is called a *crossing point* if one of these two paths goes through *white* half-edges issuing out of this vertex.

X3. If two cycles in an X-graph have only one common vertex which is a crossing point, then there exist two non self-intersecting cycles with the same property.

X-graph is called X*-planar* if it is can be drawn without self-intersections in the plane so that in the process of going around each vertex black and white half-edges alternate.

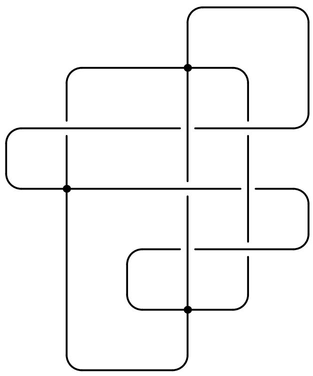

Figure xgraf.

X4. (a) An X-graph on fig. xgraf is X-planar.

(b) Is any planar X-graph also X-planar?

X5. Using Jordan theorem it can easily be proved that *if two cycles in an X-graph have only one common vertex which is a crossing point, then this X-graph is not X-planar.*

(a) If among the common vertices of two not self-intersecting cycles in an X-graph exactly one is a crossing point, then the X-graph is not X-planar.

(b)* **The X-planarity Criterion.** An X-graph is X-planar if and only if it does not contain two non self-intersecting cycles without common edges and having exactly one crossing point.

PROBLEMS SUGGESTED AT INTERMEDIATE CONSIDERATION.

After intermediate consideration we accept soltions of problems 1b, 4b, 4c, X2, X5.

Realization of graphs with cross-structures.

X6. (a) If an X-graph contains two cycles with an odd number of crossing points, then the X-graph contains two cycles with one crossing point.

(b) If two cycles of an X-graph have one common point which is crossing point, then the X-graph is not X-planar.

(c) If among the common vertices of two cycles of an X-graph there are odd crossing points, then this X-graph is not X-planar.

X7. A path in an X-graph is *turning*, if by going through each point along this path the colour of half-edge changes. Consider a turning Euler cycle T in some X-graph.

(a) Clearly, if we go along cycle T, then we pass through every vertex of X-graph twice. If we pass through some vertex by half-edges of *different* colours, then there are two cycles with exactly one crossing point.

(b) Suppose that the chord diagram of the cycle T is as shown on fig. diagr. Then the X-graph contains two cycles with exactly one crossing point.

(c) If the chord diagram of the cycle T is not planar then the X-graph contains two cycles with exactly one crossing point.

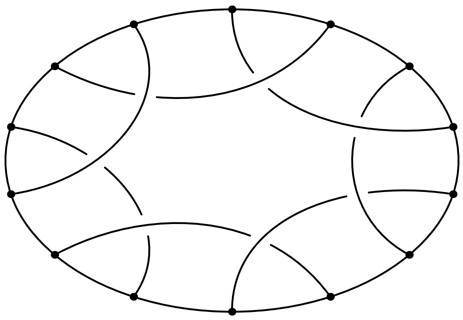

Figure diagr

Recognition algorithm for genus of a graph.

5. Any graph can be drawn without self-intersections

(a) in the 3-space.

(b) on the sphere with certain numbers of handles (fig. sferag) depending on the graph.

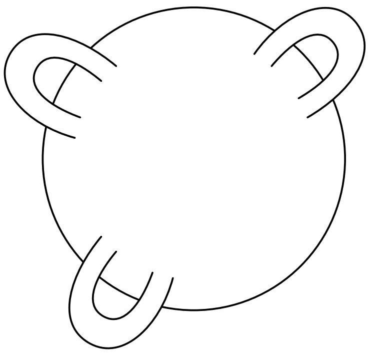

Figure sferag

The smallest number g for which graph G is embedable in the sphere with g handles is called *the (orientable) genus* of G.

6. Find the genus of K_5.

The Euler formula for the torus. *Suppose that on the torus a connected graph is drawn without self-intersections. Let V be the number of its vertices and E the number of its edges. A face is a close domain on the torus which is bounded by edges of the graph. Denote by F the number of faces. Suppose that any face has a unique boundary circle and is split by any closed polygonal line (i.e. any face is topologicaly equivalent to the disc in the plane). Then $V - E + F = 0$.*

7. (a) Graph K_8 could not be embedded in the torus.

(b) For any sphere with handles there is a graph which could not be embedded there.

The Euler formula for the sphere with g handles. *Suppose that on the on the sphere with g handles a connected graph is drawn without self-intersections. Let V be the number of its vertices and E the number of its edges. A face is a close domain on the on the sphere with handles which is bounded by edges of the graph. Denote by F the number of faces. Suppose that any face has a unique boundary circle and is split by any closed polygonal line (i.e. any face is topologicaly equivalent to the disc in the plane). Then $V - E + F = 2 - 2g$.*

Fix an orientation on sphere with g handles. An R-graph is called *R-embeddable* in sphere with g handles, if it can be drawn without self-intersections in the sphere with g handles so that for any vertex the cyclic ordering of edges in the clockwise direction around this vertex coincides with the prescribed cyclic ordering.

The smallest number g for which R-graph is R-embedable in the sphere with g handles is called *the (orientable) genus* of R-graph.

8. (a) Find the genus of any R-graph corresponding to graph K_4.

(b) Find an algorithm for recognition the genus of R-graph using the Euler formula for the sphere with g handles.

(c) Find an algorithm for recognition the genus of graph.

Embedability of graphs with systems of rotations in the Möbius strip.

9.* (a) Is it possible to cut out disjoint Möbius strip and a cylinder from the Möbius strip?

(b) Is it possible to cut out two disjoint Möbius strips from the Möbius strip?

A *graph with rotations* (shortly: *an UR-graph*) is defined as a graph with

1) a disposition of zeroes and units on its edges, and,

2) for any vertex, an orientable cyclic ordering of half-edges issuing out of the vertex.

(Remark. In fact, in mathematics another definition is used: a graph with rotations is the class of the above-mentioned objects up to equivalence defined below).

For UR-graphs the notions of UR-subgraph, being UR-homeomorphic etc. are defined analogously to ordinary graphs.

10. (a) Consider K_4 with numbered vertices. How many disposition of zeroes and units on edges together with, for every vertex, orientable cyclic orderings of the half-edges issuing from this vertex are there (i. e. how many UR-graphs correspond to the ordinary graph K_4 with numbered vertices)?

(b)* The same question for non numbered vertices (i. e. how many UR-graps up to UR-isomorphism correspond to the ordinary graph K_4).

An UR-graph is called *UR-embeddable into the Möbius strip,* if the UR-graph could be drawn without self-intersections in the Möbius strip, and small oriented circles around vertices could be added so that the following propeties hold:

(1) for any vertex the cyclic ordering of edges in the oriented circle around this vertex coincides with the fixed cyclic ordering.

(2) an edge contains zero if and only if the orientations of the circles around the ends of the edge *agree along this edge* (fig. ori).

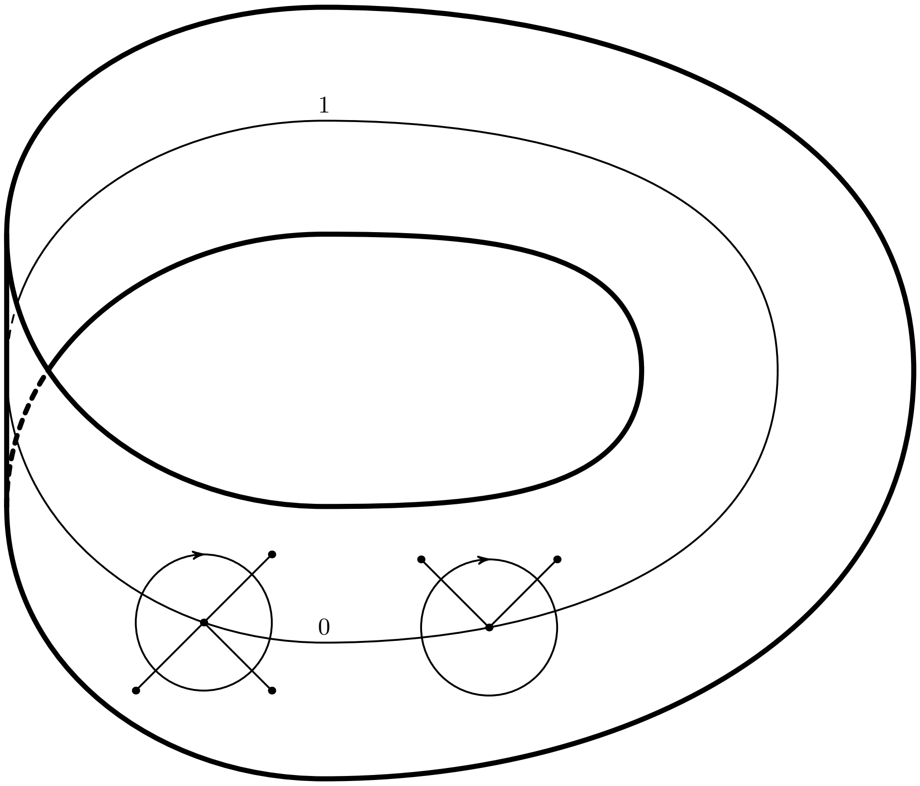

Figure ori

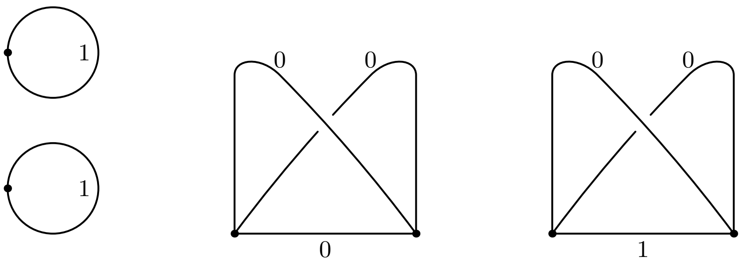

Figure ootheta

11. (a) Which of the UR-graphs corresponding to the graph K_4 are UR-embeddable into the Möbius strip?

(b) The UR-graphs on fig. ootheta are not UR-embeddable into the Möbius strip.

(c) Does there exist an UR-graph non-UR-embeddable into the Möbius strip and non-containing UR-subgraphs, UR-homeomorphic to one of the UR-graphs on fig. ootheta?

(d) An *inversion at a vertex a* is changing the orientation of cycle of half-edges issuing out of a, together with simultaneous substitution of zeroes by units and units by zeroes on all edges issuing out of a. Two UR-graphs are called *equivalent* if one of them can be obtained from the other using inversions (possibly at different vertices). Prove that equivalent UR-graphs are simultaneously UR-embeddable into the Möbius strip or not.

(e) Find an analogue of the R-planarity criterion for UR-embeddability of UR-wedges into the Möbius strip.

(f) If by deleting some edge from an UR-graph we obtain a tree, then the UR-graph is UR-embeddable into the Möbius strip.

12.* (a) **Theorem on embeddability of cubic UR-graphs into the Möbius strip.** A cubic UR-graph is UR-embeddable into the Möbius strip if and only if the UR-graph does not contain an UR-subgraph which is UR-homeomorphic to an UR-graph equivalent to one of those shown on fig. oothera.

(b) Find the analogue of the theorem of embeddability of cubic UR-graphs into the Möbius strip for arbitrary UR-graphs.

13.* A *graph with non-orientable rotations* (shortly: *U-graph*) is a graph for each of whose vertices a (non-ordered) cyclic ordering of the edges issuing from this vertex is fixed.

State and prove analogues of the above problems for U-graphs.

SOLUTIONS AND HINTS TO PROBLEMS OFFERED AT INTERMEDIATE CONSIDERATION.

Am Ende konnte ja jede gut gelöste
Mathematikaufgabe geistigen Genuss bringen...
H. Hesse, Das Glasperlenspiel.

0. (a) The Jordan theorem could be used. We present more elegant (but in fact analogous to the above) proof of the non-planarity of K_5 based on the Euler formula.

Suppose that K_5 is drawn without self-intersections in the plane. Then by the Euler formula $5 - 10 + F = 2$. Hence $F = 7$. Put two arrows on each edge of K_5 drawn in the plane — one to the left and the other to the right. Then the number of arrows is equal to $2E = 20$. Since the boundary of any face consists of at least three edges, it follows that the number of arrows is at least $3F = 21 > 20$. Contradiction.

Using this idea one can prove the non-planarity of graph $K_{3,3}$.

(b) **Statement:** for any planar connected graph with at least three vertices, without loops and multiple edges, $2E \geq 3F$ and $E \leq 3V - 6$.

Solution. Put two arrows on each edge of the graph drawn in plane — one to the left side and other to the right. Then the number of arrows is equal to $2E$. Since every face contains at least three arrows, it follows that $2E \geq 3F$. Thus $6 = 3(V - E + F) \leq 3V - E$. Therefore $E \leq 3V - 6$.

(c) If degree of every vertex is at least 6, then $2E \geq 6V$. This contradicts to (b).

(d) Suppose that a polyhedron contains neither vertices of degree 3 nor triangular faces. Since this is a polyhedron, it follows that degree of every vertex at least 4. So $2E \geq 4V$. Since there are no triangular faces, it follows that every face contains at least 4 edges. Therefore $2E \geq 4F$. So $2 = V - E + F \leq 0$. Contradiction.

1. (a) The answer is $16 = 2^4$. Hint: since from every vertex of a graph three edges are issuing, it follows that there exist two different orientable cyclic ordering of half-edges issuing from the vertex.

(b) Hint: the cyclic orderings of half-edges issuing from vertices could be changed in 0, 1, 2, 3, 4 vertices. Hence there are at most 5 non-R-isomorphic R-graphs.

2. (a) Hint: consider each graph from problem 1(a) taking symmetry into account.

(b), (c) Hint: induction (cutting out a hanging vertex).

(d) Join the vertex A with A_1 and B with B_1 by paths in the tree. If these paths intersect precisely in one point, then the R-graph contains can R-subgraph R-homeomorphic the 'eight' figure (see fig. 8theta). If these paths intersects precisely by an arc, then the original R-graph contains R-subgraph R-homeomorphic to the 'theta' figure (see fig. 8theta). Other cases reduce to those considered above.

(e) Suppose that an R-graph does not contain R-subgraphs R-homeomorphic to one of shown on the fig. 8theta. Make the construction from (d). Then from (d) it follows that if vertices A, A_1 lie on one red edge and B, B_1 lie on the other red edge, then vertices A, A_1, B, B_1 lie on the boundary of the polygon in the same (or in the inverse) order. Hence we can connect this vertices inside the polygon by disjoint segments. So given R-graph is R-planar.

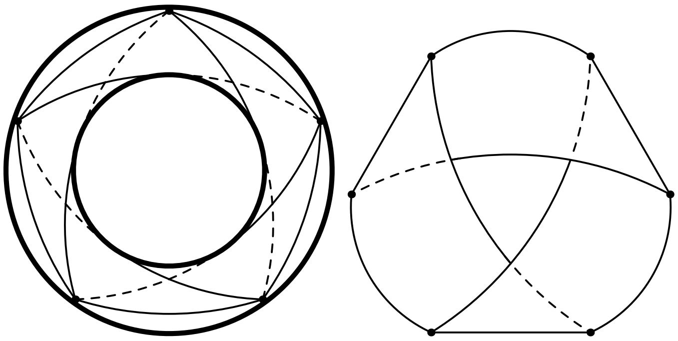

Figure real

3. (a) Yes, it is. See fig. real.

(b), (c) In the sphere it is not possible. Suppose to the contrary that graph K_5 is drawn without self-intersections in the sphere. If we delete a point (which is not a vertex of K_5) from the sphere, then we obtain a plane. But K_5 is not planar.

Since K_5 is not planar and the cylinder (the annulus) could be drawn in the plane without self-intersections it follows that the cylinder (the annulus) it is not possible.

(d) It is possible. See fig. real.

X1. (a) The necessity. Suppose that a chord diagram is drawn in the plane without self-intersections. Then the union of edges, which are not chords, splits the plane into two parts. Put in one set the chords which lie in one part, and put in the other set the chords which lie in the other part.

The sufficiency. Draw the cycle $S(T)$ in the plane without self-intersections. Draw chords of the first set as broken lines inside the cycle. They can be drawn without intersections because the corresponding chords are not skew. Analogously chords of the second set can be drawn without intersections outside the cycle. So we embedded the chord diagram in the plane.

(b) The sufficiency. Suppose that a chord diagram of some Euler cycle is planar. Contract in the plane each chord to a point (fig. point) We obtain an embedding of the graph in the plane.

The necessity. Start to move from some vertex A through an arbitrary edge. We go out of each vertex by the edge which belongs to the same face as the edge through which we came in this vertex. We go through each edge no more than once. We shall move in such way as long as possible. The path cannot be continued only at A. Moreover we returned to A by an edge belonging to the same face as the starting edge (otherwise we can continue the motion). So we have obtained a cycle F, going through each vertex along edges belonging to one face (including A, as was shown earlier). If F is not an Euler cycle, then from some of its vertices B there issue some edges not belonging to F. Move from B through the cycle F and then continue the movement in the same way as while constructing F. As a result we obtain a larger cycle going through each vertex along edges belonging to one face. Thus the cycle can be enlarged until it would contain all the edges of the graph.

Let us prove that the chord diagram of the obtained Euler cycle T is planar. Split the Euler cycle T at each vertex and connect in the plane the split vertices by chords (fig. chord). We obtain in the plane a chord diagram of the cycle T.

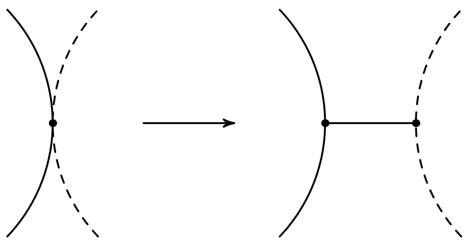

Figure chord

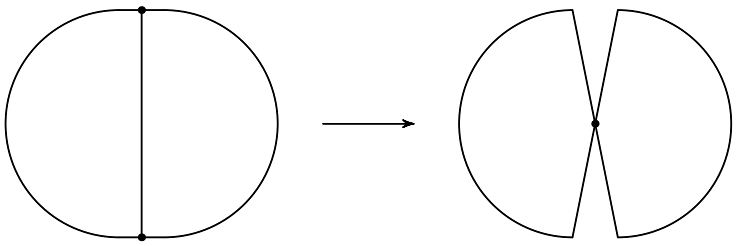

Figure point

X3. Suppose that K_1 and K_2 are the two cycles, having exactly one crossing point A. Take a cycle K_1^+ without self-intersections which consists of edges of the cycle K_1 and contains the vertex A. Analogously define a cycle K_2^+. Thus K_1^+ and K_2^+ are the required cycles.

X4.

(a) See fig. realxgraf.

(b) No. The X-graph "eight" (fig. eight) is planar, but is not X-planar.

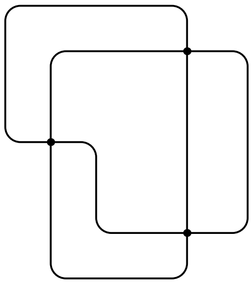

Figure realxgraf

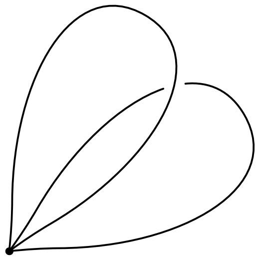

Figure eight

SOLUTIONS AND HINTS TO THE REMAINING PROBLEMS.

Realization of graphs with cross-structure. See [Per05].

Realization of graphs with rotations.

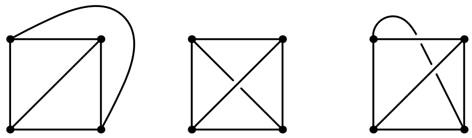

Figure 1.

1. (b) The answer is 3, see fig. 1 (proved by N. Goncharuk, S. Krupin and G. Pogudin).

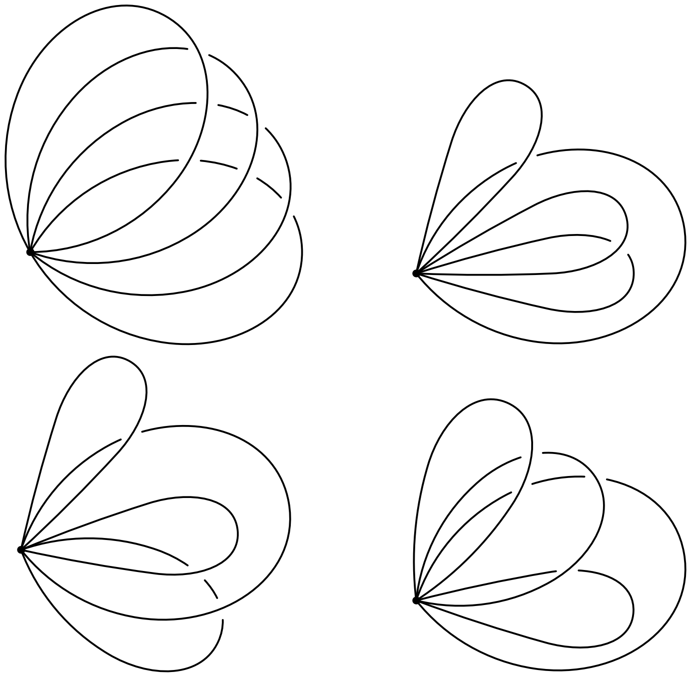

Figure 2.

4. (b) The prohibited R-subwedges are shown in fig. 2.

(б) Conjecture: the prohibited R-subgraphs are shown in fig. 3 (at the end).

Recognition algorithm for genus of a graph.

5. (a) Draw a graph (maybe with intersections) in the plane. If there are intersection points, then lift a part of one of the every two intersecting edges in the 3-space so that intersection vanish.

(b) Hint: use the idea of solution of (a).

6. The answer is 1.

7. (a) Analogously to 0.a.

(b) Analogously to 0.a the graph K_{2g^2+9} is not embeddable in the sphere with g handles.

8. (a) The answer is 0, 1, 1 for the three graphs shown in fig. 1, respectively.

(c) Hint: use (b), considering each R-graph, corresponding to the given graph.

(b) The following notion of *thickening* appears in many problems of topology and its applications. We use this construction for recognition the genus of a graph.

For a graph G consider disjoint union of disks, the number of disks equals to the number of vertices of the graph G. On the boundary circle of each disk mark disjoint intervals corresponding to the edges issuing from the corresponding vertex. For every edge of the graph G join (maybe not in the plane) two intervals with the strip corresponding to this edge. Notice that this could be done in two ways (fig. 4). Let N be the union of constructed disks and strips (N is a 2-manifold). The pair (N, G) consisting of N and the graph G is called a *(two dimensional) thickening* of G. Sometimes the 2-manifold N alone is called thickening. (A graph G is called a *spine* of the 2-manifold N.)

Figure 4.

Examples of thickenings. (1) The cylinder with the middle line and the Möbius strip with the middle line (fig. 5) are thickenings of the circle.

Figure 5.

(2) If a graph is embedded in the plane (or into a 2-manifold) then we can chose a neigborhood of a graph, and the neighborhood is a thickening of this graph.

(3) If we have a the general position map of a graph G in the plane (or in a 2-manifold) then we can construct a thickening of G corresponding of the map (fig. 6).

Figure 6.

(4) An example of a thickening of $K_{3,3}$ is shown in fig. 7. The nubmber of boundary circles equals to 3. From the Euler formula it follows that the thickening is homeomorphic to the torus with the hole.

Figure 7.

A thickening is called *orientable*, if it is possible to orient the boundary circles of disks so that for any strip the arrows on two opposite sides, lying on the oriented circles, have opposite directions along the strip.

The answer to 8b. The orientable genus of a graph G equals to

$\frac{1}{2}(2 - V + E) - \frac{1}{2}\max\{h(N) \mid N \text{is an orientable thickening of the graph } G\}$.
Hint: for an orientable thickening patching all boundary circles we obtain a sphere with handles.
Clearly, that there is a combinatorial algorithm for calculation of the number $h(N)$ (prove!).

Embedability of graphs with systems of rotations in the Möbius strip.

9.* (a) Yes, it is possible.
(b) No, it is impossible.

10. (a) The answer is $2^4 \cdot 2^6 = 2^{10}$.
(b)* We do not have a solution.

11. (a) We do not have a solution.

(b) Suppose that some of the UR-graphs is UR-emnedded in the Möbius stip. The neighborhood of the first UR-graph is the union of the two Möbius stips. If we make a cut perpendicular to an edge of the third UR-graph, then we obtain the union of two Möbius stips. The proof of the non-embeddability of the second UR-graph is analogous to the proof of the non-embeddability of this UR-graph in the plane.

(c) Yes. Use the inversion at a vertex of the second or the third graph in fig. ootheta.

(d) Suppose that some UR-graph is UR-embedded in the Möbius stip. Change the direction on some circle. We obtain the embedding of an UR-graph which differes from the initial UR-graph by the invertion at one vertex.

(e) We do not have a solution.

(f) Draw the tree inside a polygon so that the vertices of the tree would be vertices of the polygon. The remaining edge can be drawn outside.

12.* (a) See [Per].
(b) This is an unsolved problem.

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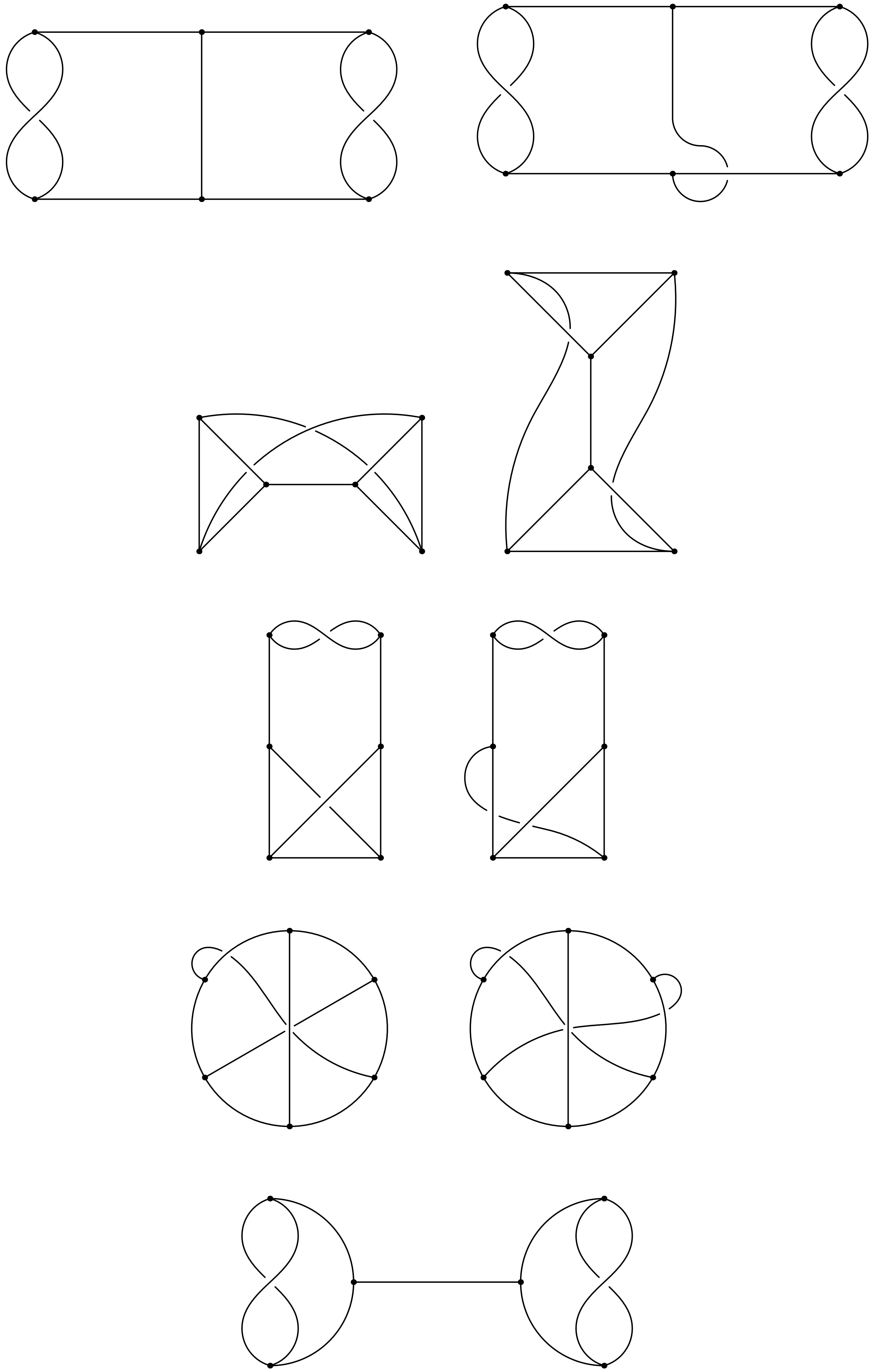

Figure 3.