Robbers sharing boxed loot

I.Bogdanov, G.Chelnokov, E.Kulikov

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Problems

***i* (An aside problem).** Two greedy robbers try to share the loot. The World War III will come soon, so each wants to obtain a part with a maximal price in dollars as well as in euro. Assume that each robber can estimate precisely (by his own opinion) each part of the loot in dollars and in euro. These estimates may not to be proportional, and estimates by different robbers may differ. Moreover, the robber does not know his partner's estimating function.

Each robber will be satisfied if he obtains a part which costs a minimum a half of a whole price in dollars and a half in euro (estimated by his own rules). Prove that there does not exist an algorithm possessing such sharing. (On each step of an algorithm one of the robbers divides one of the parts of the loot into two parts having specified prices, if possible. If the parts can be distributed between robbers so that each one has the desired quantity, then the sharing is done.)

The result of the next problem implies that such sharing exists, though there is no finite algorithm to find it.

1. a) Each robber will be satisfied if the price of his part will be one cent less than a half (or more, of course) of a common price. Prove that the robbers may obtain the desired sharing.

b) The same question about five currencies (for instance, dinars, yens and yuans).

c) The same question for five robbers (each wants one cent less than 1/5 of the whole price in each currency).

Warning! You may need to use some problems FROM BELOW in solving the first one!

2. The robbers have 301 boxes including gold and rum.

a) Cpt. Flint may take any 151 boxes. Prove that he may choose them in order to obtain half of the whole amount of gold and half of the whole amount of rum. Show that the number 151 cannot be changed by 150 in this problem.

b) Find the least possible number Flint is allowed to take if he is always capable to take 1/3 of gold and 1/3 of rum.

c) The same question for 2/5 instead of 1/3.

d) The same question, if Flint is capable to take 2/5 of each resource, and the number of boxes is 303.

3abc. The same questions for 3 types of resources (gold, rum and pirate maps).

While solving problem 3 the next problem is very useful:

4. Suppose there are some boxes, each containing not more that 1 kg of gold and 1 liter of rum.

a) Suppose the number of boxes is even. Then one can share the boxes into two parts with equal number of boxes so that the total amount of gold differs not more than by 1 kg, and the total amount of rum differs not more than by 1 l.

b) Analogous question about the division into 3 (4, 5, etc.)parts.

The general formulation. Suppose there are N boxes containing resources of k different types. Cpt. Flint wants to take some of them so that the share of each resource in his boxes is α or more.

Denote by $f_{N,k}(\alpha)$ the minimal number of boxes Flint is sufficient to take. For brevity, we will write $f_N(\alpha)$ instead of $f_{N,k}(\alpha)$, if there is no chance of misunderstanding.

5. a) Prove that there exists a constant c_k depending on k (but not on N) satisfying the following property. Suppose that each box contains not more that 1 kg of each resource. Then one can share the boxes into two parts so that the amount of each resource in parts differs not more than by c_k kg.

b) Prove that there exists a constant d_k (depending on k only) such that $f_{N,k}(1/2) \le N/2 + d_k$, i.e., that Flint can always choose $N/2 + d_k$ boxes out of N to obtain a half of each of k resources.

c) The same question as in a) for 3 (4, 5, etc.) parts.

d) Prove that there exists a constant $d_{k,m}$ (depending on k and m only) such that $f_{N,k}(1/m) \le N/m + d_{k,m}$.

6 (Greedy Flint). In this problem we want to obtain a bound which is valid for all N **and is the best possible for infinite set of values of** N.

In this problem, partial results like those in prob. 5 will also be submitted.

a) N boxes contain gold and rum, with exactly 1 kg of the whole weight in each box. Prove that Flint can take all the gold from some boxes and all the rum from the others so that the amount of gold remaining in boxes will not exceed $(N+1)/4$ kg, and the amount of remaining rum also will not exceed $(N+1)/4$ kg.

b) N boxes contain resources of 4 types, with exactly 1 kg of the whole weight in each box. Flint wants to take from each box the resources of only two types so that the amount of any resource left will not exceed α kg. What is the least α for which Flint will always be satisfied?

c) Find the answer, if the number of resources is k, and Flint is allowed to take all but one resource from each box.

d) Find the answer, if the number of resources is k, and Flint is allowed to take resources of only ℓ types from each box.

7. Let c_k be the constant defined in problem 5a). **a)** Prove that $c_k \le 2k-2$.

b) Prove that $c_k \le k$.

c) Prove that $c_k \le (k+1)/2$.

8. a) Suppose there are 99 boxes with gold, rum and maps. Prove that Flint can choose either 34 boxes with 1/3 of each resource amount, or 67 boxes with 2/3 of each resource amount.

b) Suppose there are 1000 boxes with resources of 5 types. Prove that Flint can choose either 201 boxes with 1/5 of each resource amount, or 803 boxes with 4/5 of each resource amount.

For convenience, we will use the notation $s = k-1$ (recall that k is the number of types). Denote by $\lceil x \rceil$ the least integer which is not less than x, and by $\lfloor x \rfloor$ the greatest integer which is not greater than x.

9. a) Prove that for any sufficiently large N

$$f_N(1/2) = s + \left\lceil \frac{N-s}{2} \right\rceil.$$

b) Prove that for any sufficiently large N

$$f_N(1/3) = s + \left\lceil \frac{N-s}{3} \right\rceil.$$

c) Prove that for any sufficiently large N

$$f_N(2/3) = s + \left\lceil \frac{2N-s}{3} \right\rceil.$$

d) Prove that for any p,q there exists $N(p,q)$ such that for any $N \ge N(p,q)$

$$f_N(p/q) = s + \left\lceil \frac{pN-s}{q} \right\rceil.$$

Advice. The solutions of 9abc can be written for some fixed (sufficiently large, with appropriate residue modulo q) value of N, so that the text was not overloaded by formulas.

10†: Open questions and conjectures. a) For any N, k, α there exists an example of situation, where $f_{N,k}(\alpha)$ boxes if necessary to take, and each box containonly one type of resources.

b) What is $f_N(\alpha)$ when N is not sufficiently large? And/or α is irrational?

c) How can be strengthened the result of problem 7? (One of the authors believes in the bound of order $\sqrt{k}$, another — even in const$\cdot \ln k$.)

Intermediate finish

11. a) Robbers have stolen N boxes with k types of fruits. Flint wants to take exactly half of fruits of each type. Find the minimum number of boxes he needs to unpack.

b) What is the answer if he wants to get exactly $d\%$ of fruits of each type?

c) While Flint was solving the former problem, his Grandma made jam of fruits in each box and put it back to its box. Prove that the answer in the former problem does not change.

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Solutions

i. At first, let us reformulate our problem in a more convenient way. Assume that the desired algorithm of sharing exists. Then on each step one of the robbers divides one of the parts into two ones. As one robber knows nothing about estimation functions of the second robber, we can choose how much do these new parts cost on the second's mind (the only condition to be satisfied is that the sum of the prices of two parts should be equal to the whole price). Our goal is to choose these prices so that the desired distribution will not be reached after the current turn. If we can do this on each turn, then algorithm is not valid.

Stop and realize it…

Let us explain how do we assign the values. Assume that the value of the whole loot is \$100 and 100 euro from each robber's mind.

We will always choose other robber's values of the parts to satisfy the following condition: overall value (value in euro plus value in dollars) of each part must be equal for both robbers. Assume that the robbers reach the goal. Then the first one takes the part with overall value at least 100. Then this part has also overall value of at least 100 in the second robber's mind. So the other part has overall value of at most 100 in the second robber's mind. That means that both parts have overall value exactly 100, and then the value of exactly 50 euro and exactly 50 dollars for both robbers.

Suppose the robbers could not share the loot before current step, but the sharing becomes possible when one of them divides some part into two parts A and B. It is obvious that parts A and B will be taken by different robbers (otherwise the sharing was possible before this step).

Let us show that we can choose the values of A and B according to previous rules in such a way that sharing is impossible as before. The number of ways to assign second's values to parts is infinite. Assume that each of these ways allows the sharing. Then there are two different assignments when the parts taken by first robber (including A) are exactly the same (since there are finite ways to share parts between two robbers) and so parts taken by second robber (including B) are the same too. But the value of B in these two cases is different so in one of them the second robber does not obtain exactly \$50 and 50 euro. Contradiction.

1a). Let the first robber divide all the loot into 101 parts with prices equal to $1/101$ in dollars as well as in euro. Then the second robber can choose 51 of this parts with at least half of euro value and at least half of dollar value in the terms of his estimation function (one can use the variant of problem 2a) to prove it). The first robber will thus obtain exactly $50/101 > 49/100$ in the terms of his estimated function.

Remark. Another solution can be based on problem like in the next part. In this solution robbers may perform only much simpler actions (see remark below 1b)).

1b), 1c). We will use problem 5c).

Suppose that we share the treasure into boxes in some way. Introduce a resource for any pair robber–currency. So the quantity of some resource in the box is the cost of treasure in this box in terms of estimated function of this robber using this currency. So we will have 10 (or 25 in part c)) resources.

If the quantity of each resource in any box does not exceed x, then boxes can be distributed among 5 groups with difference of quantities of any resorse not exceeding xc_{25} (c_25 can be obtained from 5a) for part b) and from 5c) (for 5 groups) for part c)). Let us choose such n that c_{25}/n does not exceed one cent, and let us share all the loot into boxes with the condition that the quantity of each resource in any box does not exceed $1/n$. We can perform that in the following manner. At first let the first robber to share all the treasure equally (by each currency) between n boxes. The let the second robber to divide each box equally (by each currency) into n parts. We obtain n^2 boxes with values not exceeding $1/n$ by any estimated function of first and second robbers. (In problem c) he third, the fourth and the fifth robbers perform the similar operations). We have got the collection of boxes that we needs.

Now, if every robber declares his value of each box in each currency and they will use algorithm from 5, then they will reach the goal.

Remark. Note that robbers must not share boxes *equally*. It is sufficient just to divide all the loot into parts with quantity of each resource not exceeding $1/n$. It can be done using only much simplier actions.

2a). Obviously, if all the boxes contain equal quantities of all the resources, then Flint should take at least 151 boxes.

Let x_i be the quantity of gold in ith box. Let us sort boxes in descending order by quantity of gold (e.g. $x_1 \geq x_2 \geq \dots \geq x_{301}$). Suppose we share the boxes with numbers from 2 to 301 into two groups with 150 boxes in them so that the quantity of gold in both groups differs by at most x_1. Then the problem is solved since we can choose the group with the greater quantity of rum and add the first box into it. We present two ways of such division.

Way 1. Let us take boxes with numbers 2, 4, 6, ..., 300 into the former group and boxes with numbers 3, 5, ..., 301 — into the latter one. Then the quantity of gold in the former group is not less than the quantity of gold in the latter; on the other hand, the quantity of gold in former group with second box eliminated is not more than the quantity of gold in the later group. So the quantities of gold in groups differ at most by $x_2 \leq x_1$.

Way 2. Let us share boxes in any way. If quantities of gold in groups differ by more than x_2 then we take a box with the most quantity of gold from a group containing more gold (group A) and exchange it with a box with the least quantity of gold from the other group (group B) (obviously, the quantity of gold in the first box is more than the quantity of gold in the second box). Note that we cannot get the situation where the quantity of gold in group B is greater than the quantity of gold in group A by at least x_2 after this operation. But the quantity of gold in group A becomes smaller and smaller. As there are only finite ways of sharing boxes into two groups, this process will stop, and we will reach the goal.

2b). Answer. 101 boxes.

One can easily see that if the resources are shared into boxes equally then 100 boxes is not sufficient. We will present an algorithm which allows us to take 101 boxes.

Analogously to the previous solution, the goal will be reached if we share the boxes with numbers from 2 to 301 into three groups so that the difference of quantities of gold in any two of them is not more than x_1.

Way 1. Boxes with numbers of 2, 5, 8, ..., 98 we will put into the first group, those with numbers of 3, 6, ..., 99 — into the second one, and those with numbers of 4, 7, ..., 100 — into the third one.

Way 2. Let us share the boxes in any way. If the quantity of gold in some two groups differs by more than x_2, we will exchange a box with maximum quantity of gold from the group with the maximum quantity of gold with a box with minimum quantity of gold from the group with minimum quantity of gold. Then the difference between the maximum quantity of gold and the minimum quantity of gold becomes smaller. Repeating such operation we will reach the goal.

2c). Answer. 122 boxes.

If we share all the rum equally into 3 boxes and share all the gold equally into other 298 boxes, then we must take at least 2 rum boxes and at least 120 gold boxes. So the answer is at least 122.

Let us prove that if we initially have 302 boxes, then the answer is still 122. Let us take two boxes with maximal quantities of gold (denote these quantities by $x_1 \geq x_2$) and share other 300 boxes into 5 groups with condition that the quantities of gold in any two groups differ by at most x_2 (as in preceding parts of this problem). We will then take two groups with maximum quantity of rum and initial two boxes with maximal quantities of gold reaching the goal.

2d). Answer. Also 122 boxes.

The example is of course the same. It remains to prove the bound.

Let us define the *process of reduction*.

Assume that there exists a box A with x_A gold and y_A rum and a box B with $x_B < x_A$ gold and $y_B < y_A$ rum. Then we will change this two boxes by the boxes A' with x_A gold and y_B rum and B' with x_B gold and y_A rum. Suppose we can choose from the new collection a set of 122 boxes which satisfies the conditions of problem; then we could do it with the old collection too. Actually, if we must take only one of the boxes A' or B' then we should take the box A instead and we will reach the goal. And if we must take both of them or none of them, then nothing changes at all.

Obviously, by a finite sequence of such reductions we will get the collection with the property that if a box X contains more gold than a box Y, then it contains not more quantity of rum than in Y.

Such collection will be referred to as "reducted collection".

So it is sufficient to prove the statement of the problem only for reducted collections.

Let us sort boxes in descending order by quantity of gold in them (in case of equality — in ascending order by quantity of rum). Then if we will take boxes with numbers 1, 3, 6, 8, 11, 13, ..., 296, 298, 301, 303 we will obtain at least 2/5 of gold. Actually, it is easy to see that we will take at least 2/5 from any

5-tuple of boxes with numbers $5k-4$, ..., $5k$. But if we will renumber them in reverse order, the boxes chosen will have the same numbers. That means that we will obtain at least 2/5 of rum.

Remark 1. Other parts of this problem can be solved similary.

Remark 2. It is sufficient to take boxes in such way that for any k among the first k boxes we have at least $\frac{2}{5}k$ chosen (this guarantees that in boxes chosen we have at least $\frac{2}{5}$ gold) and among the last k boxes we have at least $\frac{2}{5}k$ chosen (this guarantees that in chosen boxes we have at least $\frac{2}{5}$ rum). One can prove that a bound obtained in this way is exact for any rational α if the number of boxes is sufficiently large.

3a). Answer. 152 boxes.

We will refer to resources as gold, rum and maps.

In both solutions we will show that we can choose 152 boxes even from 302 boxes.

Example. Let us put all the gold into one box, all the rum into other one, and distribute all maps equally between all other boxes.

Bound 1. As we declared we use problem 4a) as a lemma.

Let us choose the box containing the maximum amount of gold and the box that has maximum amount of rum between other ones. Then, by problem 4a), we can share all other boxes into two groups (each consisting of 150 boxes) with the condition that the difference between quantities of gold in groups is not more than the quantity of gold in the first chosen box, and the difference between quantities of rum in groups is not more than the quantity of rum in the second chosen box. The we add these two boxes to the group having the maximum quantity of maps. Obviously, this collection of 152 boxes satisfies conditions of problem.

Bound 2. We will use the next theorem.

The sandwich theorem. Suppose we have three bodies in space (e. g. broad, cheese, and ham). Then there exists a plane that divides each of these three bodies into two equal by volume parts.

Let us mark 302 points in general position in space (i. e. such that no 4 of them lies on the same plane). Then, if we put a VERY little box in every point, then no plane intersects 4 boxes. (Actually, we can take boxes with diameter $d/3$, where d is the minimum distance from a marked point to the plane on which some three others lie.)

By the sandwich theorem, one can divide gold, rum and maps (really, such a sandwich) by one plane into two parts equal by volume. This plane intersects not more than 3 boxes. Hence at least 150 boxes lie on one side of this plane (we will refer to it as "left" side). Let us take all the boxes but these left ones; then we get the desired collection since they will contain all the "right" part in them.

3b). Answer. 102 boxes — no matter, among 301 or 302.

Example. The same as in the previous part.

Bound. We can use the scheme similar to the first solution of the previous part. Let us choose the box that contains the most quantity of gold (assume that it contains a gold) and then the box that contains the most quantity of rum among remaining boxes (assume that it contains b rum). One can share the other boxes by 4b) between three groups with 100 boxes in each so that the following condition is satisfied: quantities of gold in any two groups differ by not more than a, and quantities of rum in any two groups differ by not more than b. So let us add first two boxes into the group that contains the greater quantity of maps. Well, in this group there lies at least third part of gold, rum and maps.

3c). Answer. 122 boxes.

Jury does not know an elementary solution of this problem without using methods of 9d).

4a). Denote the number of boxes by $2n$. Let a_i and b_i be quantities of gold and rum in ith box.

Proof 1. Let us prove the statemnt of the problem by induction on n. The base $n=0$ is obvious. Let us prove the induction step.

Let us take two of $2n+2$ boxes with maximum quantities of gold. They will be referred to as the first and the second boxes ($a_1 \geq a_2$). By induction hypotesis one can share other boxes into two groups with n boxes in each so that the conditions of our problem are satisfied. Denote the quantities of gold in groups by a and a', and quantities of rum in groups by b and b'. Then $|a-a'| \leq a_2$ (since $a_i \leq a_2$ for $3 \leq i \leq 2n+2$) and $|b-b'| \leq \max_i b_i$. Without loss of generality, $b \leq b'$. Suppose $b_1 \leq b_2$. Then put the first box into the second group and the second box into the first group. Sums of a_i in groups become $a+a_2$ and $a'+a_1$; we have $|(a+a_2)-(a'+a_1)| \leq |a-a'|+|a_2-a_1| \leq a_2+a_1-a_2 = \max_i a_i$. Next, sums of b_i in groups become $b+b_2$ and $b'+b_1$; then we have $|(b+b_2)-(b'+b_1)| \leq \max(b'-b, b_2-b_1) \leq \max_i b_i$ because b_2-b_1 and $b-b'$ have different signs. So this sharing satisfies the conditions of the problem. Conversely, if $b_1 \geq b_2$, then place the first box into the first group and the second box into the second group. One can see by the similar caculations that $|(b+b_2)-(b'+b_1)| \le \max_i b_i$ and $|(a+a_2)-(a'+a_1)| \le a_1$.

Proof 2. Sort the boxes by the quantity of gold in descending order (i.e. $a_1 \ge \ldots \ge a_{2n}$). We will call *a pair* two boxes with numbers $2i-1$ and $2i$. If we share the boxes into two groups with condition that the boxes from any pair will be shared into different groups, then quantities of gold in groups differ by at most $(a_1-a_2)+(a_3-a_4)+\ldots+(a_{2n-1}-a_{2n}) \le a_1$.

Let us share the pairs into groups in any way. Suppose that we have not reach the goal and the quantity of rum in the first group is much more than in the second one. Then there exists a pair where quantity of rum that comes to the first group is more than that going to the second. Let us exchange boxes of this group. Actually, the difference of the quantities of rum decreases by absolute value since it changes by at most $2b_1$. This process cannot continue infinitely, so we will reach the goal.

4b). We can launch a process similar to that in previous part. We will present it for the sharing boxes into five groups; other numbers of groups can be done analogicaly. We suppose that the number of boxes is divisible by 5.

Sort the boxes by the quantity of gold in descending order (i.e. $a_1 \ge \ldots \ge a_{5n}$). We will call *a 5-tuple* the boxes with numbers $5i-4$, $\ldots$, $5i$. Suppose we distribute boxes into five groups so that the boxes of any 5-tuple are shared into different groups; then quantities of gold in any two groups differ by at most $(a_1-a_5)+(a_6-a_{10})+\ldots+(a_{5n-4}-a_{5n}) \le a_1$.

Let us distribute all the 5-tuples into groups in any way. Assume that the quantity of rum in the first groups is maximal (denote it by b), while this quantity in the second group is minimal (denote it by b'). If $b-b' \le b_1$ then we reach the goal. Suppose $b-b' > b_1$. Then there exists a 5-tuple in which the quantity of rum that goes to the first group is more than the quantity of rum that goes to the second group. Let us exchange these two boxes; then the quantity of rum in the first group decreases (by at most b_1) and the quantity of rum in the second group increases. Then the difference of quantities of rum in these two groups decreases by absolute value. Besides, the quantity of rum in first group is still more than b', while that in the second one is still less than b.

So we have two possibilities — the maximum quantity of rum in groups decreases, or it remains the same but the number of groups that contain such quantity decreased. So this process cannot continue infinitely, and we will reach the goal.

5a). The main idea is a development of that of problem 4. Sort the boxes in ascending order by the quantity of the first resource. Let us combine the first and the second boxes, the third and the fourth boxes and so on into pairs (these pairs will be called as the pairs of the first rank). If the number of boxes is odd then let us add an empty box. In further text we will consider only those distributions in which the boxes of one pair are distributed into different groups. Then in any such distribution quantities of the first resource in groups differ by not more than the quantity of gold in the first box, and hence by not more than 1 kg.

Next, we will percept a first rank pair as a box. In each pair a box is called to be "bigger" if it contains not less quantity of second resource as other box (in case of equality we will call any of them (but just one) the "bigger" one). We will say that we "take" some pair into a group if we take "bigger" box into this group (hence taking the other box into another group). Words "a pair A contains x of ith resource" means that the difference between quantities of ith resource in "bigger" and other box is x. Note that this number may be negative.

Sort pairs in descending order by the quantity of the second resource, and then combine the first and the second pairs, the third and the fourth pairs and so on into the pairs of the second rank. Analogously, we will consider only such distributions where the pairs of the first rank lying in the same pair of the second rank are taken into different groups.

This restriction guarantees that the quantities of the first and the second resource in groups differ by at most 1 kg. Besides, the quantity of each resource in the second rank pair (definited in a similar way) is at most 2 kg by absolute value.

We can analogously define the pairs of the ith rank for $i=1,\ldots,k$ (we combine into dth pair of the ith rank two pairs of the $(i-1)$th rank having the numbers $2d-1$ and $2d$ in descending order by ith resource). The quantity of each resource in any ith rank pair is at most 2^{i-1} kg by absolute value, and for any distribution of pairs of the ith rank the difference between quantities of any of first i resources will not exceed 2^{i-1}. So any distribution of the kth rank pairs satisfies conditions of the problem.

5b). Suppose that for each resource Flint takes c_k boxes with maximum quantity of this resource (c_k is defined in 5a)). Then we can suppose that the quantity of any resource in corresponding box chosen is at least 1 kg while that in all remaining boxes is at most 1 kg. Using 5a) we can distribute boxes into two groups with the difference between quantities of any resource not exceeding c_k, so Flint will take a group with minimum quantity of boxes.

Using this algorithm Flint will take not less than half of each resource with at most $k\cdot c_k+\dfrac{N-k\cdot c_k}{2}=N/2+\text{const}$ boxes.

5c). The solution is similar to the solution of 5a); the only change is that instead of the ith rank pairs we must collect all boxes into the ith rank k-tuples (of course each of them consists of k tuples of rank $(i-1)$).

5d). This solution is similar to solution of 5b).

6a). Sort boxes by the quantity of gold in ascending order. Denote by α_i the weight of gold in ith box ($\alpha_1\le\alpha_2\le\ldots\le\alpha_N$). If $\alpha_1+\alpha_2+\ldots+\alpha_N\le(N+1)/4$, then we can throw all the rum away. Otherwise there exists k such that

$$\alpha_1+\ldots+\alpha_k\le\frac{N+1}{4} \tag{6a.1}$$

and

$$\alpha_1+\ldots+\alpha_{k+1}>\frac{N+1}{4}. \tag{6a.2}$$

Denote $\alpha=\alpha_{k+1}$. From (6a.2) we see that $(k+1)\alpha>(N+1)/4$. Let us throw away rum from the first k boxes and gold from all other. Using (6a.1) we see that quantity of gold at most $(N+1)/4$. Note that the boxes with numbers $\ge k+1$ contain not more than $1-\alpha$ kg of rum. So there is at most $(N-k)(1-\alpha)$ kg of rum remained. Introduce the following notation: $N'=N+1$, $k'=k+1$. Then we are to prove that $(N'-k')(1-\alpha)\le N'/4$. But $1-\alpha<1-N'/(4k')$. So it is sufficient to prove that

$$\frac{(N'-k')(4k'-N')}{4k'}\le\frac{N'}{4}.$$

By series of reduction we got

$$N'^2+4k'^2\ge 4N'k',$$

which is obvious.

Remark. The bound given is exact only for odd N. Evidently, it can be shown that the exact bound for even N is $\dfrac{N(N+2)}{4(N+1)}$.

Solution 2. This solution is little longer than the first, but it is generalizable.

Denote by α_i and $\beta_i=1-\alpha_i$ the quantities of gold and rum in the ith box. Actually, we must prove that the following system has an integer solution:

$$x_1+y_1=1,\quad \ldots,\quad x_N+y_N=1, \tag{6a.3}$$

$$x_1\alpha_1+\ldots+x_N\alpha_N\le\frac{N+1}{4},\quad y_1\beta_1+\ldots+y_N\beta_N\le\frac{N+1}{4}, \tag{6a.4}$$

$$0\le x_i,y_i\le 1. \tag{6a.5}$$

If $x_i=0$ then we must take gold from ith box, while if $x_i=1$ we must take rum. Informally speaking, x_i is part of gold that we must remain in the ith box.

Well, if we need a solution without condition that all x_i's are integer, then we can make one with $N/4$ in right parts of (6a.4) instead of $(N+1)/4$. Actually, let

$$x=\frac{N}{4(\alpha_1+\cdots+\alpha_N)},\quad y=\frac{N}{4(\beta_1+\ldots+\beta_N)}.$$

Then

$$x+y\ge\frac{4}{1/x+1/y}=1.$$

So we have a solution $x_1=\cdots=x_N=x$, $y_1=\cdots=y_N=1-x\le y$ (if $x>1$ then let $x=1$, $y=0$). It will be useful to arrange the variables as a table with N coloumns and two rows (in the first row x_i's are placed, while in the second y_i are). We will call equations and inequalities by *horizontal* and *vertical* conditions. We will change our solution (i. e. the set of values of our indeterminates), but it will always remain a solution.

At first, suppose that there exist two columns with non-integer values (wlog the first and the second ones). Let us try to add some numbers to the values in these columns in the way that the values of all expressions in the system will not change except of the left part of first inequality (which can only decrease). Let us add ε to x_1; then we must substract ε from y_1. Then we will add $c\varepsilon$ to y_2 to keep the horizontal value unchanged (actually, c is β_1/β_2, but it does not matter). At the end we must substract $c\varepsilon$ from x_2. Then the value in the first row changes by $d\varepsilon$ where d is a constant not depending on ε. So by choosing the sign of ε we will not increase this value. Besides, if ε is sufficiently small, then all the variables will remain to lie on [0,1].

Huh-h-h-h...

Now we can choose such ε that one of variables becomes integer, and all variables remain to lie on [0;1] (Another value in this column will also make integer). Then the number of number of "non-integer" columns is decreased. Using this operations we will obtain a solution of our system (with the same substitution of right parts in horizontal inequalities) with non-integer values in just one column (let it be the first column). Now we will try to change x_1 with 1 and y_1 with 0 or x_1 with 0 and y_1 with 1. In the first case the first horizontal expression will increase by $(1-x_1)\alpha_1$, while in the second case the second expression will increase by $(1-y_1)\beta_1 = x_1(1-\alpha_1)$. The product of these two expression does not exceed 1/16 so one of them does not exceed 1/4. After such changing original system will be satisfied.

Remark. This estimation is exact only for odd N. We think that exact value for even N is $\dfrac{N(N+2)}{4(N+1)}$.

6b). Jury does not know the full solution of this problem. We will prove bounds that are similar to the one in 5. Namely, we will prove that a constant in statement satisfies the inequalities $\dfrac{N}{8} \le c(N) \le \dfrac{N}{8}+1$.

Example. Let us share all resources equally between all boxes. Then we will have at least half of one of the resources remained, e.g. at least $N/8$ of this resource. Estimation is not exact for any value of N.

Bound. Let us make the same trick as in the second solution of the previous part. Actually, we construct a system of linear equations and inequalities with variables x_i, y_i, z_i, t_i; the value of such variables means which part of the corresponding resource are we to leave in the ith box. The system has the following form:

$$x_1+y_1+z_1+t_1=2, \quad \dots, \quad x_N+y_N+z_N+t_N=2, \tag{6b.1}$$

$$x_1\alpha_1+\ldots+x_N\alpha_N \le c(N), \quad \dots, \quad t_1\delta_1+\ldots+t_N\delta_N \le c(N), \tag{6b.2}$$

$$0 \le x_i, y_i, z_i, t_i \le 1. \tag{6b.3}$$

We are to find an appropriate value of $c(N)$ for which this system has an integer solution.

If $c=N/8$ we will always have non-integer solution. Really, let

$$x=\frac{N}{8(\alpha_1+\cdots+\alpha_N)}, \quad \dots, \quad t=\frac{N}{8(\delta_1+\ldots+\delta_N)}, \qquad x+y+z+t \ge \frac{16}{1/x+1/y+1/z+1/t}=2.$$

We will have a solution $x_i=x$, ..., $t_i=t$ (we have $x_i+y_i+z_i+t_i \ge 2$, hence if this inequality is strict, then we will decrease all variables by the same number — all the inequalities remain valid). Let us change the values of our variables. Analogously, we will arrange our variables as a table $4\times N$.

We will describe the process that allows us to obtain a solution of our system having sufficiently large number of integer values.

Consider the following graph. Its vertices are the rows of our table, and two vertices are connected by an edge if there is a column with both of the variables in these rows being non-integer (two vertices can be connected by several edges). If a column contains more than two non-integer numbers (e. g. a_1, ..., a_k), then we will connect only rows with no non-integer variables between them in current column (e.g. we will connect only $a_1-a_2-\cdots-a_k$).

Suppose this graph contains a cycle. Let us take any of its edges. It corresponds to a pair of non-integer numbers in the same column. Let us add ε_1 to one of them and substract it from the other. Let us take the second edge of the cycle and add ε_2 to one of its variables and substract it from the other (we must do it in such way that the value of the left part of inequality in common row of these edges remains unchange). Then ε_2 will be proportional to ε_1. In the samy way we will find ε_3, ε_4 and so on. Then left parts of inequalities in all but one row will remain the same. Now we can choose the sign of ε_1 so that the remaining left part will not increase, and all variables remain to lie on [0,1], and at least one becomes integer.

We can repeat this operation until we obtain a graph having no cycles.

Then our graph is forest, hence we have a vertex with rank one. So only one of the values in that row is not integer. We can increase it and decrease the value on another end of this edge. Then the right part of this row in (6b.2) will increase at most by 1. Now we can continue our process forgetting this vertex. At the end we will increase any right parts by not more than 1. So we have an integer solution with $c(N)=N/8+1$.

6c). Answer. $\dfrac{N+k-1}{k^2}$.

Example. Share all resources equally between all boxes and let $N=mk+1$. Then after reduction we will have at least $m+1$ boxes with some resource, and so we will have at least $\dfrac{m+1}{k}=\dfrac{N+k-1}{k^2}$ kg of this resource remained.

Bound. We use induction by k. The base is problem . Let us prove the induction step.

Suppose that we have $k+1$ type of resources. Let us call the first resource as gold. Enumerate boxes in ascending order by the quantity of gold. Let α_i denote the quantity of gold in ith box ($\alpha_1 \le \alpha_2 \le \ldots \le \alpha_N$). If $\alpha_1+\ldots+\alpha_N \le \dfrac{N+k}{(k+1)^2}$, then one can take all resources but gold from each box. Otherwise there exists t that satisfies the unequalities

$$\alpha_1+\ldots+\alpha_t \le \frac{N+k}{(k+1)^2} \tag{6c.1}$$

and

$$\alpha_1+\ldots+\alpha_{t+1} > \frac{N+k}{(k+1)^2}. \tag{6c.2}$$

Then let us leave gold in first t boxes and take it from all other boxes. From (6c.1) we obtain that the quantity of gold will not exceed $\dfrac{N+k}{(k+1)^2}$. Denote $\alpha=\alpha_{t+1}$. Then using (6c.2) we obtain

$$(t+1)\alpha > \frac{N+k}{(k+1)^2}. \tag{6c.3}$$

But there is at most $1-\alpha$ kg of resources remained in any box with the number $\ge t+1$. Using induction hypotesis we can leave in each box just one resource with condition that the quantity of each resource does not exceed $c=(1-\alpha)\dfrac{(N-t)+k-1}{k^2}$. It remains to prove that $c\le \dfrac{N+k}{(k+1)^2}$.

Using condition (6c.3) obtain

$$c \le \left(1-\frac{N+k}{(k+1)^2(t+1)}\right)\frac{(N-t)+k-1}{k^2}.$$

After substitutions $N'=N+k$ and $t'=t+1$ we have

$$((k+1)^2t'-N')(N'-t') \le k^2t'N'.$$

After expanding brackets this inequality makes $N'^2+(k+1)^2t'^2 \ge 2(k+1)N't'$ which is obvious.

6d). Jury again does not know the full solution of this problem. Denote the exact bound in this problem by c. We will prove that $N/(4\ell)\le c \le N/(4\ell)+1$ if $k\ge 2\ell$, and $(k-\ell)N/k^2 \le c \le (k-\ell)N/k^2+1$ otherwise. It is possible to prove more exact estimations but it will use a lot of paper for nonsignificant purpose.

We will continue using ideas of 6b): we construct a system of linear equations and inequalities with variables x_{id}; the value of x_{id} denotes which part of the dth resource we must reamin in the ith box. The quantity of the dth resource in the ith box will be denoted by α_{id}.

The system has the following form:

$$x_{11}+x_{12}+\cdots+x_{1k}=k-\ell, \quad \ldots, \quad x_{N1}+x_{N2}\cdots+x_{Nk}=k-\ell, \tag{6d.1}$$

$$x_{11}\alpha_{11}+x_{21}\alpha_{21}+\ldots+x_{N1}\alpha_{N1}\le c, \quad \ldots, \quad x_{1k}\alpha_{1k}+x_{2k}\alpha_{2k}+\ldots+x_{Nk}\alpha_{Nk}\le c, \tag{6d.2}$$

$$0\le x_{id}\le 1. \tag{6d.3}$$

We want to estimate such values of c for which system have integer solution. At first we want to know for what c it has any solution. Actually, we will find such c that our system has solution of the form $x_{1d}=x_{2d}=\ldots=x_{Nd}=t_d$.

Let $S_d=\sum_i \alpha_{id}$ (we have $\sum_d S_d=\sum_{i,d}\alpha_{id}=N$). Then one obtains the following conditions on t_d:

$$t_d \le \min\left(1, \frac{c}{S_d}\right). \tag{6d.4}$$

If there exists such t_d that inequalities (6d.4) is satisfied and $\sum_d t_d = k-\ell$, then the solution of original system is found.

So we must find such c that such t_d always exist. This is the case when

$$\sum_d \min\left(1, \frac{c}{S_d}\right) \geq k-\ell \tag{6d.5}$$

for any S_d such that $\sum_d S_d = N$. Let r be the number of values of d with condition $S_d \leq c$. Then the number of unities in our sum is also r, and our inequality looks like

$$c \geq \frac{k-\ell-r}{\sum\limits_{d=1}^{k-r} \frac{1}{S_d}},$$

with condition that $\sum\limits_{d=1}^{k-r} S_d \leq N$. Let us estimate the denominator of expression in right part:

$$\sum_{d=1}^{k-r} \frac{1}{S_d} \geq \frac{(k-r)^2}{\sum\limits_{d=1}^{k-r} S_d} \geq \frac{(k-r)^2}{N};$$

so this expression does not exceed $\dfrac{(k-\ell-r)N}{(k-r)^2}$. Hence it is sufficient to choose

$$c \geq \max_{r=0,\ldots,k-\ell} \frac{(k-\ell-r)N}{(k-r)^2}.$$

Let us denote this maximum by B and find it. This maximum is reached when the following expression tends to minimum:

$$\frac{(k-r)^2}{k-r-\ell} = (k-r-\ell) + \frac{\ell^2}{k-r-\ell} + 2\ell.$$

The last operand is constant, while the product of first two operands is also a constant. So their sum takes the least value when these terms are possibly closer. Hence, if $k-\ell \geq \dfrac{\ell^2}{k-\ell}$ (e.g. $k \geq 2\ell$) then maximum is reached when $k-r-\ell = \dfrac{\ell^2}{k-r-\ell}$, e.g. $r = k-2\ell$; otherwise it is reached when $r=0$.

So if $k \geq 2\ell$, then $B = N/(4\ell)$ else $B = (k-\ell)N/k^2$. Within this value of c we will have solution (but maybe non-integer).

So if we change a solution as in , then the graph introduced makes a forest. Again let us use the process of "truncating" of values from the same problem. We will have each constant increased by at most 1. So we have proved that for $c = B+1$ the system (6d.1)–(6d.3) has an integer solution, QED.

Example. We share first $k-r$ resources between boxes equally, and we choose not use other $k-r$ resources at all. Then we have at least

$$\frac{N}{k-r} - \frac{N\ell}{(k-r)^2} = \frac{N(k-r-\ell)}{(k-r)^2} = B.$$

of some resource remained. This means that B is lower bound. So we prove that $B \leq c \leq B+1$.

Some general words on systems of linear equations and inequalities.

As far as we are going to investigate solutions of systems of linear equations and inequalities, we think it is important to enlist some general methods of working with them.

A system of linear equations and inequalities has the form

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n = c_k \\ \hline \end{array} \tag{$*$}$$

$$b_{11}x_1+b_{12}x_2+\ldots+b_{1n}x_n \le d_1$$
$$\vdots \qquad (**)$$
$$b_{l1}x_1+b_{l2}x_2+\ldots+b_{ln}x_n \le d_l$$

We assume that the set of solutions of this system is bounded, e. g. there exists some C such that for any solution of system the inequality $|x_i| \le C$ is satisfied. Besides, we assume that the system has at least one solution.

At first we will reduct system $(*)$ using the Gauss method to the staircase form:

$$\begin{array}{ccccccccccccc}
a'_{1,i_1}x_{i_1} & + & a'_{1,i_1+1}x_{i_1+1} & + & a'_{1,i_1+2}x_{i_1+2} & + & a'_{1,i_1+3}x_{i_1+3} & + & \ldots & + & a'_{1n}x_n & = & c'_1 \\
 & & a'_{2,i_2}x_{i_2} & + & a'_{2,i_2+1}x_{i_2+1} & + & a'_{2,i_2+2}x_{i_2+2} & + & \ldots & + & a'_{2n}x_n & = & c'_2 \\
 & & & \ddots & & & \vdots & & & & & & \vdots \\
 & & & & a'_{k,i_k}x_{i_k} & + & a'_{k,i_k+1}x_{i_k+1} & + & \ldots & + & a'_{kn}x_n & = & c'_k
\end{array}$$
$$(1 \le i_1 < \ldots < i_k)$$

Then we can express $x_{i_1}, \ldots, x_{i_k}$ via other variables using these equations and substitute these variables with there expressions in inequalities $(**)$. After this operation we will have the system with the less number of variables consisting only of inequalities. Note that we eliminate not more than k variables (less number of variables can disappear if some equations become $0=0$).

Consider any solution of our system (it exists by our conditions) and let us change this solution in the following way. Suppose that none of inequalities becomes equality. Then we can take one of variables and begin to change continuously its value (in any direction). After some time one of inequalities must become an equality since the solution set is bounded.

So, we just proved that there is some solution of this system where some inequality turns into the equality. So we will further consider the new system where this inequality is changed by the corresponding equality (and stands in the system $(*)$ instead of $(**)$). Then, applying the Gauss method, we eliminate one more variable (or one more equation makes $0=0$).

Obviously we can continue this process while there exists at least one variable that cannot be expressed via other variables and constants. Finally, all the variables become constants.

So we got the solution of initial system in wich at least $n-k$ inequalities become equalities. This is the result we will use in the following.

We apply this method to prove next theorem. (By k-dimensional vector we will call just a ordered tuplet of k numbers; vectors can be added and multiplied by a number component-wise).

The Caratheodory theorem. Suppose we have n vectors in k-dimensional space $\vec{\mathbf{u}}_1,\ldots,\vec{\mathbf{u}}_n$, and vector $\vec{\mathbf{v}}$ lies in there convex hull, e. g. there exist such $\alpha_i \ge 0$, that $\alpha_1+\cdots+\alpha_n=1$ and

$$\vec{\mathbf{v}} = \alpha_1\vec{\mathbf{u}}_1+\cdots+\alpha_n\vec{\mathbf{u}}_n. \qquad (***)$$

Then there exist $k+1$ vectors among $\vec{\mathbf{u}}_i$'s such that $\vec{\mathbf{v}}$ lies in their convex hull.

Proof. Condition $(***)$ is a system of k linear equations with n variables α_i. We add condition $\alpha_1+\cdots+\alpha_n=1$ and inequalities $\alpha_i \ge 0$. Then there exists a solution of this system where $n-k-1$ of this inequalities becomes equalities, e.g. with at most $k+1$ non-zero coeffecients. But it means that $\vec{\mathbf{v}}$ lies in the convex hull of all vectors having nonzero coefficients.

7a), 7b). This time we will assign variable to each box: value $x_i=1$ means that the ith box in first group and $x_i=-1$ — in the second. Let's denote the quantity of the dth resource in the ith box by α_{di}. Let us try to distribute the boxes with condition that quantity of all resources in both groups is equal:

$$\alpha_{11}x_1+\alpha_{12}x_2+\cdots+\alpha_{1N}x_N=0$$
$$\vdots \qquad (7a.1)$$
$$\alpha_{k1}x_1+\alpha_{22}x_2+\cdots+\alpha_{kN}x_N=0$$
$$|x_1|\le 1 \quad \ldots \quad |x_N|\le 1$$

(each inequality $|s|\le t$ is just the two linear inequalities: $-t\le s$ and $s\le t$). One solution is already exists — $x_i=0$. Using techniques that were described above we'll find solution with $N-k$ inequalities turned into equalities e. g. $N-k$ variables are ±1 (we'll call other variables *bad*). Let us substitute positive bad variables with ones and negative — with -1. Then to each of the right parts in (7a.1) we will add at most $k\cdot 1$ kg and substract at most $k\cdot 1$ kg. So its absolute valus will not exceed k.

Remark. Other solution of this problem can be done using multidimmensional variant of the Sandwich Theorem (in k-dimensional space plane can divide volume of k bodies in equal parts). But proof of this theorem is very non-elementary.

7c). We will sharpen the bound of the previous solution. Suppose that bad variables have numbers from 1 to k. Suppose that $y_i=(x_i+1)/2$. By 6a) problem we can share numbers from 1 to k into two subsets A and B with condition that $\sum\limits_{i\in A} y_i \le (k+1)/4$ and $\sum\limits_{i\in B}(1-y_i)\le(k+1)/4$. If $i\in A$ then let us change x_1 to -1 else to 1. Each left part will be increased at most by $(N+1)/2$ and decreased at most by $(N+1)/2$. But all vaiables now is integer. So we reach the goal.

8a). Consider 99-tuples of numbers $(x_1,x_2,\ldots,x_{99})$ with conditions

$$x_i\in[0,1],\quad \sum_i x_i\vec{\mathbf{u}}_i=(1/3,1/3,1/3)=\frac{1}{3}(1,1,1).$$

This is a system of linear restrictions on x_i with three equations among them. So there exists a solution with at least 96 inequalities making exact, e. g. 96 of its coordinates are integer.

There are two possibilities: there exist 65 coordinates equal to zero or 32 coordinates equal to unity (otherwise we have at most $64+31=95<96$ integer coordinates). If 65 coordinates is zero (wlog they are the last 65 coordinates), then

$$\sum_{i=1}^{34}\vec{\mathbf{u}}_i\ge\sum_{i=1}^{34}x_i\vec{\mathbf{u}}_i=\sum_{i=1}^{99}x_i\vec{\mathbf{u}}_i=\frac{1}{3}(1,1,\ldots,1),$$

so these (at most 34) boxes with nonzero coordinates contain at least the third part of each resource. Otherwise some 32 coordinates is 1 (wlog they are the last 32 coordinates), hence

$$\sum_{i=1}^{67}\vec{\mathbf{u}}_i\ge\sum_{i=1}^{67}(1-x_i)\vec{\mathbf{u}}_i=\sum_{i=1}^{99}(1-x_i)\vec{\mathbf{u}}_i=\frac{2}{3}(1,1,\ldots,1),$$

and we have at least two thirds of each resource in 67 boxes having coordinates not equal to 1.

8b). As in the previous part, consider 1000-tuples of numbers $(x_1,x_2,\ldots,x_{1000})$ for which

$$x_i\in[0,1],\quad \sum_i x_i\vec{\mathbf{u}}_i=(1/5,\ldots,1/5)=\frac{1}{5}(1,\ldots,1).$$

This is the system of linear restrictions on x_i with five equations. So there exists a solution with 995 inequalities becoming equalities, e. g. 995 coordinates are integer.

We can proceed absolutely similarly to the previous part (we have 799 zeros or 197 unities). So by repeating ideas of the previous part we get the solution.

9. In all examples each box will contain fruits of only one type and all fruits of the ith type will be shared equally between boxes that contain this type. So all examples will be written in the form $(a_1,a_2,\ldots,a_t)$ where a_i is the number of boxes with the ith type of fruits.

Correspond to the ith box a k-dimensional vector $\vec{\mathbf{u}}_i=[u_{i1},\ldots,u_{ik}]$ with the coordinate u_{ij} being equal to the quantity of the jth resource in the ith box. We suppose that our vectors are normed, e. g. $\sum_i\vec{\mathbf{u}}_i=[1,1,\ldots,1]$ (we can easily obtain this by appropriate choice of the measuring standard for each resource).

9a). Example. $(1,1,\ldots,1,N-k+1)$.

Bound. Consider N-tuples of numbers $(x_1,\ldots,x_N)$ for which

$$x_i\in[0,1],\quad \sum_i x_i\vec{\mathbf{u}}_i=(1/2,1/2,\ldots,1/2)=\frac{1}{2}(1,1,\ldots,1).$$

This is system of linear restrictions for x_i with k equations among them. So there exists a solution with $n-k$ inequalities making exact, e.g. with $n-k$ coordinates being integer.

There are two cases: $N-f_N(1/2)$ of coordinates are zero or

$$f_N(1/2)-s=\left\lceil\frac{1}{2}(N-s)\right\rceil\geq\left\lfloor\frac{1}{2}(N-s)\right\rfloor=N-f_N(1/2)$$

of coordinates are 1. Otherwise number of integer coordinates does not exceed

$$(N-f_N(1/2)-1)+(f_N(1/2)-s-1)=N-k-1$$

which is impossible.

Wlog the coordinates chosen are the last $N-f_N(1/2)$ coordinates (we change $A=(x_1,\dots,x_N)$ by $(1-x_1,\dots,1-x_N)$ if necessary). Suppose that $x_{f_N(1/2)+1}=x_{f_N(1/2)+2}=\dots=x_N=0$. Then

$$\sum_{i=1}^{f_N(1/2)}\vec{\mathbf{u}}_i\geq\sum_{i=1}^{f_N(1/2)}x_i\vec{\mathbf{u}}_i=\sum_{i=1}^{N}x_i\vec{\mathbf{u}}_i=\frac{1}{2}(1,1,\dots,1),$$

and we reach the goal.

9b). Example. $(1,\dots,1,N-k+1)$.

Bound. Consider N-tuples of numbers $(x_1,\dots,x_N)$ for which

$$x_i\in[0,1],\qquad\sum_i x_i\vec{\mathbf{u}}_i=\frac{1}{3}(1,1,\dots,1).$$

Using similar technique as in the previous part, we get solution with at most k non-integer coordinates. So $N-f_N(1/3)$ coordinates are zero or $f_N(1/3)-s$ coordinates are 1.

In the first case the first $f_N(1/3)$ boxes give the desired collection. In the second case inverting all the values we get that the first

$$N_1=N-f_N(1/3)+s=s+\left\lfloor\frac{2}{3}(N-s)\right\rfloor$$

boxes contains at least 2/3 of all the resources. Then we can choose

$$f_{N_1}(1/2)=s+\left\lceil\frac{1}{2}(N_1-s)\right\rceil\leq s+\left\lceil\frac{1}{2}\cdot\frac{2}{3}(N-s)\right\rceil=f(1/3),$$

so that they contain at least half of resources of those contained in these $N-1$ boxes, e. gf. the third of the whole amount. We reach the goal.

9c). Example. $(2,\dots,2,N-2(k-1))$.

Bound. Consider a solution of the system

$$\left\{(x_1,\dots,x_N):\quad x_i\in[0,1],\quad\sum_i x_i\vec{\mathbf{u}}_i=\frac{2}{3}(1,1,\dots,1)\right\}.$$

with the least number of non-integer coordinates (their number is at most k). We have $N-f_N(2/3)$ zero coordinates or $f_N(2/3)-s$ unity coordinates. In the first case the first $f_N(2/3)$ boxes give the desired collection.

In the second case first $N-f_N(2/3)+s=N-N_1$ boxes contain at least a third part of all resources. So by the first part of this problem we can choose $f_{N_1}(1/2)$ boxes out of remaining $N_1=f_N(2/3)-s$ boxes with at least half of each type of resources from the remain. So we have chosen

$$N-N_1+f_{N_1}(1/2)=N-\left\lfloor\frac{1}{2}(f_N(2/3)-2s)\right\rfloor\leq N-\left\lfloor\frac{1}{2}\cdot\frac{2}{3}(N-2s)\right\rfloor=f_N(2/3)$$

boxes, ans we reach the goal.

9d). Example. Choose integer $r\in[1,q-1]$ and d such that $pr\equiv1\pmod q$, $d=\lceil pr/q\rceil$ (hence, $qd-pr=q-1$). Then if $N\geq rs$, then an example is $(r,\dots,r,N-rp)$. Actually, we must take

$$ds+\left\lceil\frac{p}{q}(N-rs)\right\rceil=\left\lceil\frac{pN+s(qd-pr)}{q}\right\rceil=\left\lceil\frac{pN+s(q-1)}{q}\right\rceil=s+\left\lceil\frac{pN-s}{q}\right\rceil,$$

boxes.

Bound. We will use induction by $p+q$. The base is proven above. Consider a solutions of the system

$$\left\{(x_1,\ldots,x_N):\quad x_i\in[0,1],\quad \sum_i x_i\vec{\mathbf{u}}_i=\frac{p}{q}(1,1,\ldots,1)\right\}.$$

One of them contains at most k non-integer coordinates. So we have $N-f_N(p/q)$ zero coordinates or $f_N(p/q)-s$ unities. In the first case the first $f_N(p/q)$ boxes make the desired collection.

In the second case we get that some $N-f_N(p/q)+s$ boxes containштп at least $(q-p)/q$ of all resources. We get two cases again: (i) $q-p>p$ и (ii) $q-p<p$.

In case (i) denote

$$N_1=N-f_N(p/q)+s=\left\lfloor\frac{(q-p)N+s}{q}\right\rfloor;$$

so

$$pN_1-s\le\frac{p}{q}((q-p)N+s)-s=\frac{q-p}{q}(pN-s).$$

We know that we can choose

$$f_{N_1}\left(\frac{p}{q-p}\right)=s+\left\lceil\frac{pN_1-s}{q-p}\right\rceil\le s+\left\lceil\frac{1}{q-p}\cdot\frac{q-p}{q}(pN-s)\right\rceil=f_N(p/q)$$

boxes out of N_1 boxes, with at least $\dfrac{p}{q-p}\cdot\dfrac{q-p}{q}=\dfrac{p}{q}$ part of resources of each type, QED.

In case (ii) we get $N-f_N(p/q)+s$ boxes contains at least $(q-p)/q$ of resources. Then among remaining

$$N_1=f_N(p/q)-s=\left\lceil\frac{pN-s}{q}\right\rceil$$

boxes we can find $f_{N_1}((2p-q)/p)$ boxes that contain at least $\alpha_1=(2p-q)/p$ of each of the resources remained. Consider these

$$k=N-f_N(p/q)+s+f_{N_1}((2p-q)/p)=N-N_1+f_{N_1}((2p-q)/p)$$

boxes. If first $N-f_N(p/q)+s$ of them contain $l_i\ge(q-p)/q$ of resources of the ith type then we obtain not less than

$$l_i+\frac{2p-q}{p}(1-l_i)=\frac{2p-q}{p}+\frac{q-p}{p}l_i\ge\frac{2p-q}{p}+\frac{(q-p)^2}{pq}=\frac{p}{q}$$

part. So we are left to prove that $k\le f_N(p/q)$. We have

$$(q-p)N_1+s\ge\frac{q-p}{q}(pN-s)+s=\frac{p}{q}((q-p)N+s),$$

so

$$k=N-N_1+f_{N_1}\left(\frac{2p-q}{p}\right)=N-s-\left\lfloor\frac{(q-p)N_1+s}{p}\right\rfloor\le N-s-\left\lfloor\frac{1}{p}\cdot\frac{p}{q}((q-p)N+s)\right\rfloor=f_N(p/q).$$

11. **Answer.** k boxes in any case.

The idea of solution is similar to ideas of 8 and 9. We just need to add that if the number of boxes is not less than the number of types of fruits, then we can fill each box using only one fruit. So we need to unpack at least one box with each type of fruits e. g. at least k boxes.