Sequences with zero sums

K. Kokhas, K. Kuyumzhiyan

Problems

1 Zero-sequences

For any nonnegative integer n let $\mathbb{Z}_n$ denote the set of residues modulo n equipped with operation "+" (addition mod n). We say that the sequence in $\mathbb{Z}_n$ is a zero-sequence if its sum equals $0 \in \mathbb{Z}_n$.

1.1. Let n be a nonnegative integer. Let k be the minimal number such that every sequence of length k in $\mathbb{Z}_n$ contains a zero-subsequence. Prove that $k = n$.

1.2. Describe all the sequences in $\mathbb{Z}_n$ of length $n-1$ that do not contain zero-subsequences.

1.3. Describe all the sequences in $\mathbb{Z}_n$ of length $n-2$ that do not contain zero-subsequences.

1.4. What is minimal m such that every sequence of length m containing at least 81 different elements of $\mathbb{Z}_{100}$ has a zero-subsequence of length 100?

1.5. In some ancient kingdom there were 4 sorts of coins that were worth several copecks each, but nobody knows the exact nominal value of these coins. What minimal number of coins should an archeologist excavate in order to be sure that he has 100 coins whose total value is an integer number of roubles?

1.6. Prove that every sequence in $\mathbb{Z}_{12}$ of length 23 contains a zero-subsequence of length 12.

1.7. Let S be a sequence of length 502 in $\mathbb{Z}_{541}$ that has exactly 10 different elements. Prove that S contains a zero-subsequence.

1.8. Let S be a sequence of length 10 in $\mathbb{Z}_{17}$ that does not contain zero-subsequences. Prove that some element of $\mathbb{Z}_{17}$ occurs in S at least 4 times.

1.9. Let S be a sequence of n integers coprime with n. Prove that every residue modulo n is a sum of some subsequence in S.

1.10. Let S be a sequence in $\mathbb{Z}_n$ of length $2n-1$. Assume that some element a occurs at least $[n/2]$ times in S. Prove that S contains a zero-sum subsequence of length n.

1.11. Let p be an odd prime number. Consider the following sequence in $\mathbb{Z}_p$: 0, 0, 1, 1, 2, 2, …, $p-1$, $p-1$. How many zero-subsequences of length p does this sequence have?

1.12. Let $n \geqslant 2$ be an integer. Let S be a sequence of length n in $\mathbb{Z}_n \setminus \{0\}$ with nonzero sum. Prove that there exist at least n different zero-subsequences in S.

2 *Extremal problems*

For every nonnegative integer n by $\mathbb{Z}_n^d$ denote the set of all arrays of the form $(m_1, m_2, \ldots, m_d)$, where each m_i is a residue modulo n. We equip this set with operation "+" (addition modulo n in each coordinate). A zero-sequence in $\mathbb{Z}_n^d$ is a sequence that has zero sum (i.e. the sequence whose sum equals $(0, 0, \ldots, 0) \in \mathbb{Z}_n^d$). By $g(n, d)$ denote the minimal number M such that in every subset of $\mathbb{Z}_n^d$ consisting of M elements there exist n elements with zero sum. By $s(n, d)$ denote the minimal number M such that in every sequence $a_1, \ldots, a_M \in \mathbb{Z}_n^d$ there exists a zero-subsequence of length n. In other words there exist different indices $i_1, \ldots, i_n$ such that $a_{i_1} + \ldots + a_{i_n} = 0$. (In fact we mean that the sequence is just a multiset, the order of elements in the sequence is not important for us.)

2.1. Prove that $s(2, d) = 2^d + 1$.

2.2. Prove that $(n-1)2^d + 1 \leqslant s(n, d) \leqslant (n-1)n^d + 1$.

2.3. Prove that $s(n_1 n_2, d) \leqslant s(n_1, d) + n_1\big(s(n_2, d) - 1\big)$.

2.4. Prove that что $g(3, 3) \geqslant 10$, $s(3, 3) \geqslant 18$. (Actually $g(3, 3) = 10$, $s(3, 3) = 19$.)

2.5. Prove that $g(n, 2) \geqslant \begin{cases} 2n-1 & \text{for odd } n; \\ 2n+1 & \text{for even } n. \end{cases}$

2.6. Consider a square 3×3 on the grid paper. Let 9 points be marked in the nodes of the grid (including the boundary of the square). Prove that the center of mass of some 4 of these points is a node of the grid too. In other words prove that $g(4, 2) = 9$.

2.7. Prove that $s(2048, d) = 2047 \cdot 2^d + 1$.

2.8. Prove that $s(432, d) = 1725$.

3 *Erdős – Ginzburg –Ziv theorem and related questions*

3.1. [Cauchy – Davenport theorem] Let p be a prime number, A and B be two nonempty subsets in $\mathbb{Z}_p$. Prove that $|A + B| \geqslant \min\{p,\ |A| + |B| - 1\}$.

3.2. [Erdős – Ginzburg –Ziv theorem] Prove that every sequence in $\mathbb{Z}_n$ of length $2n - 1$ contains a zero-subsequence.

We can use these theorems in the subsequent problems.

Many of the following problems are valid in a more general context. A commutative finite group is a finite set equipped with the operation "+" that satisfies the usual axioms for this operation: commutativity $a + b = b + a$; associativity $a + (b + c) = (a + b) + c$; existence of zero element with the property $0 + a = a$ for all a; and existence of inverse element: for any a there exists b such that $a + b = 0$ (we write $b = -a$). We can define a difference of two elements using the last axiom: $a - b$ is defined to be equal to $a + (-b)$. A typical finite commutative group looks like $\mathbb{Z}_d^n$: fix the set of numbers $k_1, \ldots, k_n$ and a set of "vectors" $(x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{Z}_{k_i}$, and the operation "+" is usual coordinatewise addition (modulo k_i for i-th coordinate).

We mark the problem by the sign † if its statement is valid for an arbitrary finite commutative group. Solutions that do not use specific properties of $\mathbb{Z}_n$ and remain valid in the case of an arbitrary finite commutative group are especially welcome.

3.3. Let p be a prime number; let $A_1, A_2, \ldots, A_k$ be nonempty subsets in $\mathbb{Z}_p$. Prove that

$$|A_1 + A_2 + \ldots + A_k| \geqslant \min\{p,\ \Big(\sum_{i=1}^{k} |A_i|\Big) - k + 1\}.$$

3.4. Let p be a prime number and let $S = (a_1, \dots, a_{2p-1})$ be a sequence in $\mathbb{Z}_p$ such that its elements $a_1, \dots, a_s$ are pairwise distinct ($s \geqslant 2$). Prove that S has a zero-subsequence of length p that contains exactly one of the elements $a_1, \dots, a_s$.

3.5. a) Take a regular 12-gon on the plane. We consider all symmetries and rotations of the plane which preserve the 12-gon. Prove that for any 47 transformations there exist 24 of them such that their composition (in some order) is an identity transformation.

b) Prove a similar statement for the symmetric group S_4.

3.6. Let p be a prime number, T be a sequence in $\mathbb{Z}_p \setminus \{0\}$ of length p, h be a maximal multiplicity of elements in T. Prove that every element of $\mathbb{Z}_p$ is a sum of at least h elements of T.

3.7. Assume $m \geqslant k \geqslant 2$, and let m be divisible by k. Prove that every sequence of integers of length $m + k - 1$ contains a subsequence of length m whose sum is divisible by k.

3.8.† [Kemperman – Scherk theorem] Let n be a nonnegative integer. Let A and B be two subsets of $\mathbb{Z}_n^d$, such that $0 \in A$, $0 \in B$ and the equation $a + b = 0$ has only the trivial solution $a = b = 0$ if $a \in A$, $b \in B$. Prove that $|A + B| \geqslant \min\{n^d,\ |A| + |B| - 1\}$.

3.9.† Let k and r be nonnegative integers and let $A = \{a_1, \dots, a_{k+r}\}$ be a sequence in $\mathbb{Z}_k$, such that 0 is not a k-sum of this sequence. Prove that the sequence has at least $r + 1$ different k-sums.

3.10.† Let S be a sequence of length n^d in $\mathbb{Z}_n^d$, let h be a maximal multiplicity of elements in this sequence. Prove that S contains a zero-subsequence of length at most h.

3.11. Let p be a prime number and $2 \leqslant k \leqslant p - 1$. Consider a sequence of length $2p - k$ in $\mathbb{Z}_p$, such that every p elements of this sequence have a nonzero sum. Prove that some element occurs in the sequence at least $p - k + 1$ times.

3.12. Let $B_1, B_2, \dots, B_h$ be a collection of subsets in $\mathbb{Z}_n^d$. Let $m_i = |B_i|$. Assume that $\sum_{i=1}^h m_i \geqslant n^d$. Prove that for each j we can choose at most one element $b_j \in B_j$ in such a way that the sum of the resulting nonempty collection of chosen elements is $0 \in \mathbb{Z}_n^d$.

3.13. Let $n > 4$ be an odd number. Let S be a sequence of length k in $\mathbb{Z}_n$, where $\frac{n+1}{2} \leqslant k \leqslant n$. Assume that S does not contain zero-subsequences. Prove that some element of $\mathbb{Z}_n$ is contained in S with multiplicity $2k - n + 1$.

3.14.† Let $A \subset \mathbb{Z}_n^d$, $|A| \geqslant n^d/k$. Prove that there exists a number r, $1 \leqslant r \leqslant k$, and a sequence $a_1, \dots, a_r$ (of not necessarily different elements of A) such that $\sum a_i = 0$.

3.15. Prove that the sequence of length $2n - 1$ in $\mathbb{Z}_n$ has a unique zero-subsequnce if and only if it consists of n copies of some number a and $(n - 1)$ copies of number b.

3.16.† Let S be a subset of $\mathbb{Z}_n^d$ containing k elements. Assume that S does not contain a subset with zero sum. Prove that there exist at least $2k - 1$ distinct elements of $\mathbb{Z}_n^d$ that can be represented as a sum of several elements of S.

3.17. Let $M = \{(a_1, b_1), (a_2, b_2), \dots, (a_{2p-1}, b_{2p-1})\}$ be a subset in $\mathbb{Z}_p^2$. Assume that every element of $\mathbb{Z}_p$ is contained in the sequence $a_1, \dots, a_{2p-1}$ at most twice. Prove that the set M contains a zero-subsequence of length p.

3.18. Prove that the minimal m such that every sequence of length m in $\mathbb{Z}_p^d$ contains a zero-subsequence is equal to $d(p - 1) + 1$.