Sequences with zero sums

I. Bogdanov, K. Kokhas, K. Kuyumzhiyan

Problems

1 *Zero-sequences*

For any nonnegative integer n let $\mathbb{Z}_n$ denote the set of residues modulo n equipped with operation "+" (addition mod n). We say that the sequence in $\mathbb{Z}_n$ is a zero-sequence if its sum equals $0 \in \mathbb{Z}_n$.

1.1. Let n be a nonnegative integer. Let k be the minimal number such that every sequence of length k in $\mathbb{Z}_n$ contains a zero-subsequence. Prove that $k = n$.

1.2. Describe all the sequences in $\mathbb{Z}_n$ of length $n-1$ that do not contain zero-subsequences.

1.3. Describe all the sequences in $\mathbb{Z}_n$ of length $n-2$ that do not contain zero-subsequences.

1.4. What is minimal m such that every sequence of length m containing at least 81 different elements of $\mathbb{Z}_{100}$ has a zero-subsequence of length 100?

1.5. Find a minimal m such that every sequence of length m in $\mathbb{Z}_{100}$, consisting of exactly 4 distinct elements, has a zero-subsequence of length 100.

1.6. Prove that every sequence in $\mathbb{Z}_{12}$ of length 23 contains a zero-subsequence of length 12.

1.7. Let S be a sequence of length 502 in $\mathbb{Z}_{541}$ that has exactly 10 different elements. Prove that S contains a zero-subsequence.

1.8. Let S be a sequence of length 10 in $\mathbb{Z}_{17}$ that does not contain zero-subsequences. Prove that some element of $\mathbb{Z}_{17}$ occurs in S at least 4 times.

1.9. Let S be a sequence of n integers coprime with n. Prove that every residue modulo n is a sum of some subsequence in S.

1.10. Let S be a sequence in $\mathbb{Z}_n$ of length $2n-1$. Assume that some element a occurs at least $[n/2]$ times in S. Prove that S contains a zero-sum subsequence of length n.

1.11. Let p be an odd prime number. Consider the following sequence in $\mathbb{Z}_p$: $0, 0, 1, 1, 2, 2, \ldots, p-1, p-1$. How many zero-subsequences of length p does this sequence have?

1.12. Let $n \geqslant 2$ be an integer. Let S be a sequence of length n in $\mathbb{Z}_n \setminus \{0\}$ with nonzero sum. Prove that there exist at least $n-1$ different zero-subsequences in S.

2 *Extremal problems*

For every nonnegative integer n by $\mathbb{Z}_n^d$ denote the set of all arrays of the form $(m_1, m_2, \ldots, m_d)$, where each m_i is a residue modulo n. We equip this set with operation "+" (addition modulo n in each coordinate). A zero-sequence in $\mathbb{Z}_n^d$ is a sequence that has zero sum (i.e. the sequence whose sum equals $(0, 0, \ldots, 0) \in \mathbb{Z}_n^d$). By $g(n,d)$ denote the minimal number M such that in every subset of $\mathbb{Z}_n^d$ consisting of M elements there exist n elements with zero sum. By $s(n,d)$ denote the minimal number M such that in every sequence $a_1, \ldots, a_M \in \mathbb{Z}_n^d$ there exists a zero-subsequence of length n. In other words there exist different indices $i_1, \ldots, i_n$, where $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant M$, such that $a_{i_1} + \ldots + a_{i_n} = 0$. Thus different sequences may coincide if considered simply as lists.

2.1. Prove that $s(2,d) = 2^d + 1$.

2.2. Prove that $(n-1)2^d + 1 \leqslant s(n,d) \leqslant (n-1)n^d + 1$.

2.3. Prove that $s(n_1 n_2, d) \leqslant s(n_1, d) + n_1\big(s(n_2, d) - 1\big)$.

2.4. Prove that что $g(3,3) \geqslant 10$, $s(3,3) \geqslant 19$. (Actually $g(3,3) = 10$, $s(3,3) = 19$.)

2.5. Prove that $g(n,2) \geqslant \begin{cases} 2n-1 & \text{for odd } n; \\ 2n+1 & \text{for even } n. \end{cases}$

2.6. Consider a square 3×3 on the grid paper. Let 9 points be marked in the nodes of the grid (including the boundary of the square). Prove that the center of mass of some 4 of these points is a node of the grid too. In other words prove that $g(4,2) = 9$.

2.7. Prove that $s(2048, d) = 2047 \cdot 2^d + 1$.

2.8. Prove that $s(432, 2) = 1725$.

3 *Erdős – Ginzburg – Ziv theorem and related questions*

3.1. [Cauchy – Davenport theorem] Let p be a prime number, A and B be two nonempty subsets in $\mathbb{Z}_p$. Prove that $|A+B| \geqslant \min\{p,\ |A|+|B|-1\}$.

3.2. [Erdős – Ginzburg – Ziv theorem] Prove that every sequence in $\mathbb{Z}_n$ of length $2n-1$ contains a zero-subsequence.

We can use these theorems in the subsequent problems.

Many of the following problems are valid in a more general context. A commutative finite group is a finite set equipped with the operation "+" that satisfies the usual axioms for this operation: commutativity $a+b=b+a$; associativity $a+(b+c)=(a+b)+c$; existence of zero element with the property $0+a=a$ for all a; and existence of inverse element: for any a there exists b such that $a+b=0$ (we write $b=-a$). We can define a difference of two elements using the last axiom: $a-b$ is defined to be equal to $a+(-b)$. A typical finite commutative group looks like $\mathbb{Z}_d^n$: fix the set of numbers $k_1, \ldots, k_n$ and a set of "vectors" $(x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{Z}_{k_i}$, and the operation "+" is usual coordinatewise addition (modulo k_i for i-th coordinate).

We mark the problem by the sign † if its statement is valid for an arbitrary finite commutative group. Solutions that do not use specific properties of $\mathbb{Z}_n$ and remain valid in the case of an arbitrary finite commutative group are especially welcome.

3.3. Let p be a prime number; let $A_1, A_2, \ldots, A_k$ be nonempty subsets in $\mathbb{Z}_p$. Prove that

$$|A_1 + A_2 + \ldots + A_k| \geqslant \min\{p,\ \Big(\sum_{i=1}^{k} |A_i|\Big) - k + 1\}.$$

3.4. Let p be a prime number and let $S = (a_1, \ldots, a_{2p-1})$ be a sequence in $\mathbb{Z}_p$ such that its elements $a_1, \ldots, a_s$ are pairwise distinct ($s \geqslant 2$). Prove that S has a zero-subsequence of length p that contains at most one of the elements $a_1, \ldots, a_s$.

3.5. a) Take a regular 12-gon on the plane. We consider all symmetries and rotations of the plane which preserve the 12-gon. Prove that for any 47 transformations there exist 24 of them such that their composition (in some order) is an identity transformation.

b) Prove a similar statement for the symmetric group S_4.

3.6. Let p be a prime number, T be a sequence in $\mathbb{Z}_p \setminus \{0\}$ of length p, h be a maximal multiplicity of elements in T. Prove that every element of $\mathbb{Z}_p$ is a sum of at most h elements of T.

3.7. Assume $m \geqslant k \geqslant 2$, and let m be divisible by k. Prove that every sequence of integers of length $m+k-1$ contains a subsequence of length m whose sum is divisible by k.

3.8.† [Kemperman – Scherk theorem] Let n be a nonnegative integer. Let A and B be two subsets of $\mathbb{Z}_n^d$, such that $0 \in A$, $0 \in B$ and the equation $a+b=0$ has only the trivial solution $a=b=0$ if $a \in A$, $b \in B$. Prove that $|A+B| \geqslant \min\{n^d,\ |A|+|B|-1\}$.

3.9.† Let k and r be nonnegative integers and let $A = \{a_1, \ldots, a_{k+r}\}$ be a sequence in $\mathbb{Z}_k$, such that 0 is not a k-sum of this sequence. Prove that the sequence has at least $r+1$ different k-sums.

3.10.† Let S be a sequence of length n^d in $\mathbb{Z}_n^d$, let h be a maximal multiplicity of elements in this sequence. Prove that S contains a zero-subsequence of length at most h.

3.11. Let p be a prime number and $2 \leqslant k \leqslant p-1$. Consider a sequence of length $2p-k$ in $\mathbb{Z}_p$, such that every p elements of this sequence have a nonzero sum. Prove that some element occurs in the sequence at least $p-k+1$ times.

3.12. Let B_1, B_2, …, B_h be a collection of subsets in $\mathbb{Z}_n^d$. Let $m_i = |B_i|$. Assume that $\sum_{i=1}^h m_i \geqslant n^d$. Prove that for each j we can choose at most one element $b_j \in B_j$ in such a way that the sum of the resulting nonempty collection of chosen elements is $0 \in \mathbb{Z}_n^d$.

3.13. Let $n > 4$ be an odd number. Let S be a sequence of length k in $\mathbb{Z}_n$, where $\frac{n+1}{2} \leqslant k \leqslant n$. Assume that S does not contain zero-subsequences. Prove that some element of $\mathbb{Z}_n$ is contained in S with multiplicity $2k - n + 1$.

3.14.† Let $A \subset \mathbb{Z}_n^d$, $|A| \geqslant n^d/k$. Prove that there exists a number r, $1 \leqslant r \leqslant k$, and a sequence a_1, …, a_r (of not necessarily different elements of A) such that $\sum a_i = 0$.

3.15. Prove that the sequence of length $2n - 1$ in $\mathbb{Z}_n$ has a unique zero-subsequnce of length n only if it consists of n copies of some number a and $(n-1)$ copies of number b.

3.16.† Let S be a subset of $\mathbb{Z}_n^d$ containing k elements. Assume that S does not contain a subset with zero sum. Prove that there exist at least $2k-1$ distinct elements of $\mathbb{Z}_n^d$ that can be represented as a sum of several elements of S.

3.17. Let $M = \{(a_1, b_1), (a_2, b_2), \ldots, (a_{2p-1}, b_{2p-1})\}$ be a subset in $\mathbb{Z}_p^2$. Assume that every element of $\mathbb{Z}_p$ is contained in the sequence a_1, …, a_{2p-1} at most twice. Prove that the set M contains a zero-subsequence of length p.

3.18. Prove that the minimal m such that every sequence of length m in $\mathbb{Z}_p^d$ contains a zero-subsequence is equal to $d(p-1)+1$.

4 Semifinal. Additional problems

2.9. Find $g(8, 2)$ and, if possible, $g(2^n, 2)$.

2.10. Prove that there exists a function $f(d)$ such that $s(n, d) \leqslant f(d)n$ for all n, d.

3.19. Prove that if $3p$ elements of $\mathbb{Z}_p^2$ with zero sum are given, then one can choose p of them with zero sum.

3.20. Let S be a sequence in $\mathbb{Z}_n$ of length at least $(n+3)/2$, $n \geqslant 3$ without zero-subsequences. Prove that S contains an element coprime with n.

3.21. Let p be a prime number. Let $A \subset \mathbb{Z}_p$ such that $|A| > 2\sqrt{p}$.
a) Prove that A contains a subset with zero sum.
b) Prove that each residue mod n can be represented as a sum of some subset of A.

5 Open questions

4.1. $g(p, 2) = 2p - 1$ for odd prime p.

For $p = 3$, 5, 7 it has been proved by Kemnitz (Kemnitz A., Exremalprobleme für Gitterpunkte. Ph.D.Thesis. Technische Universität Braunschweig, 1982), and for $p \geqslant 67$ by Gao and Thangadurai. The first step in the latter is problem 3.17.

4.2. $g(n, 2) = \begin{cases} 2n - 1 & \text{for odd } n \\ 2n + 1 & \text{for even } n \end{cases}$

4.3. $s(n, 2) = 4n - 3$.

This is Kemnitz's conjecture. It was has been proved last year by combination of algebraic and combinatorial methods. The sequence of $4n - 4$ elements without zero-subsequences of length n is the following: $(0,0)^{n-1}$, $(0,1)^{n-1}$, $(1,0)^{n-1}$, $(1,1)^{n-1}$.

4.4. Let p be a prime number, A and B be two nonempty subsets in $\mathbb{Z}_p$. Denote by $A \dotplus B$ the set of all sums of the form $a + b$, where $a \in A$, $b \in B$ и $a \neq b$. Prove that $|A \dotplus B| \geqslant \min\{p,\ |A| + |B| - 3\}$.

This statement is the conjecture of Erdős–Heilbronn. It is proved though no combinatorial proof is known.

4.5. Let S be a sequence of $4n - 4$ elements of $\mathbb{Z}_n^2$. If S does not contain a zero subsequence of length n, then S consists of 4 distinct elements with multiplicity $n - 1$ each.

This is Gao's conjecture. He has checked that this conjecture is multiplicative, i.e. if it is valid for $n = k$ and $n = \ell$, then it is valid for $n = k\ell$.

Solutions

1.1. The sequence 1, 1, ..., 1 (n units) does not contain zero-subsequences. Therefore $k \geqslant n$. Now let a_1, a_2, ..., a_n be an arbitrary sequence of length n. We prove that it contains a zero-subsequence. Consider n sums

$$\begin{aligned}&a_1\,,\\&a_1+a_2\,,\\&\dots\qquad\dots\\&a_1+\ldots+a_{n-1}\,,\\&a_1+\ldots+a_{n-1}+a_n\,.\end{aligned}$$

If neither sum is equal to zero then two sums are equal (as elements of $\mathbb{Z}_n$). Their difference determines a zero-subsequence.

1.2. Answer: these are sequences containing an element a with multliplicity $n-1$, where a coprime with n.

Indeed, let a_1, a_2, ..., a_{n-1} be a sequence that does not contain zero susequences. Suppose that $a_1 \neq a_2$. Consider n sums

$$\begin{aligned}&a_1\,,\\&a_2\,,\\&a_1+a_2\,,\\&a_1+a_2+a_3\,,\\&\dots\qquad\dots\\&a_1+\ldots+a_{n-1}\,,\\&a_1+\ldots+a_{n-1}+a_n\,.\end{aligned}$$

Since neither of the sums is equal to zero, there are two sums that coincide as elements of $\mathbb{Z}_n$. Then their difference determines a zero-subsequence. A contradiction. Therefore the sequence could not contain two different elements.

1.3. Answer: these are sequences consisting of $n-2$ copies of some element a, or sequences consisting of $n-3$ copies of some element a and one element $2a$.

We found this fact in [16].

The case $n \leqslant 5$ is left to the reader. Let the sequence contain two distinct elements $a_1 \neq a_2$. For any other element a_3 we conclude analogously to the previous solution that there are two identical elements among the elements a_1, a_2, a_1+a_3, a_2+a_3, $a_1+a_2+a_3$, ..., $a_1+\cdots+a_{n-2}$. But in this set of sums only the sums $a_1 = a_2+a_3$ or $a_2 = a_1+a_3$ may coincide, whence

$$a_3 = \pm(a_1-a_2)\,.$$

Thus for every element a_3 we have an equality $a_3 = \pm(a_1-a_2)$. Since our sequence could not contain elements that are equal to both a_1-a_2 and $-(a_1-a_2)$ (because their sum is 0), we have the same sign for all possible elements a_3 in the last equality. That is all elelemnts of the sequence except a_1 and a_2 are equal to the same number a. But $a_1 \neq a_2$, therefore one of numbers a_1, a_2, say a_1, is not equal to a. Then analogously all the elements except a_1 and a are equal, i. e. for $n > 5$ we obtain that $a_2 = \cdots = a_{n-2} = a$. And in the case $a_1 \neq a, 2a$ a zero-subsequence can be easily constructed.

1.4. Answer: $m = 102$.

Let $A = \{a_1, a_2, \ldots, a_{81}\}$ be distinct elements of some sequence. If the sequence has length 101 and the sum of all elements does not belong to A, then the sum of all elemnts except one could not be equal to 0. Now let the sequence have length at least 102. Without loss of generality we may think that the length is exactly 102. Let its sum equal to s. It is not difficult to choose two elements a_i and a_j in the set A such that $a_i + a_j = s$. Now by removing these elements we obtain a zero-subsequence of length 100.

1.5. Answer: 197 coins.

1.6. From the given sequence choose 11 pairs of numbers with even sum of numbers in each pair. Divide all the sums by 2. We obtain numbers b_1, b_2, ..., b_{11}. From these numbers choose 5 pairs with even sum of elements in each pair. Again, divide the sums by 2. We obtain numbers c_1, c_2, …, c_5. Now consider these numbers modulo 3. It is easy to see that either some resudue mod 3 has multiplicity 3 in this sequence or all 3 residues occur in it. In both cases these three residues have sum 0 mod 3. And recalling the way we have obtained the residues we obtain a desired zero-subsequence mod 12 of length 12.

1.7. This is a partial case of the theorem of Erdős and Eggleton [7]. Denote the number 502 by n, then $541 = n+39$. Let a_1, a_2, …, a_n be given sequence. Suppose that it does not contain a zero-subsequence. Fix for a moment a number m, $1 \leqslant m \leqslant n$. Observe that all the sums obtained as sums of subsequences of first m numbers of the sequence S could not be equal to sums of the form $\sum_{i=1}^{r} a_i$, where $m+1 \leqslant r \leqslant n$ (because the difference of such equal sums would determine a zero-subsequence) . We will show that it is possible to choose m in such a manner that the first m members of S will determine at least $m+39$ different sums.

Since S contains only 10 different elements, there is an element $a \neq 0$ that occurs in S at least 51 times. 541 is a prime number, hence we have the operation of division in $\mathbb{Z}_{541}$. So we can divide all elements of S by a. We will interpret the result of each division as a number from 0 to 540. So we may assume that $a_1 = a_2 = \ldots = a_{51} = 1$ and that all the other elements of S are numbers from 0 to 540. If all these numbers do not exceed 51 then every number from 1 to $\sum_{i=1}^{n} a_i$ can be represented as a sum of some a_i's. But $\sum_{i=1}^{n} a_i \geqslant 1+2+\ldots+10+(n-10) > n+39$. A contradiction.

If otherwise S contains a number greater then 51 we may assume that $a_{52} > 51$. (Observe that $a_{52} < 488$ because in the opposite case we can construct a zero-sequence containing a_{52} and several 1's). Let $m = 52$. It is obvious that the first m members of the sequence S determine at least $m+51$ sums, as we wish.

1.8. Partial case of 3.13.

1.9. Let a_1, a_2, …, a_n be a given sequence. It is trivially checked by induction that the set of sums of all subsequences a_1, a_2, …, a_i mod n contains at least i different residues mod n.

1.10. We take this proof from [16]. Let S be a given sequence of length $2n-1$. Assume $a \in \mathbb{Z}_n$ occurs $s \geqslant [n/2]$ times in S. We may also assume that $s \leqslant n-1$ because otherwise the statement is obvious.

Consider shifted sequence $S-a$, it contains zero s times. Let T_1 be a subsequence of S that consists of all nonzero elements of S, it contains at least $2n-1-s \geqslant n$ elements. By the result of the problem 1.1, we know that T_1 contains a zero-subsequence. Denote it by T_2. Assume T_2 contains t_2 elements, $2 \leqslant t_2 \leqslant n$. We may also assume that the sequence T_2 has maximal possible length (among the sequences of length at most n). If $s+t_2 \geqslant n$, then we can construct a zero-subsequence of length n by adding several 0's to T. Therefore we may assume that $s+t_2 < n$ whence $t_2 \leqslant [n/2]$ (because we still have a lot of zeroes).

Now we can easily obtain a contradiction. Since $s+t_2 < n$, it follows that $T_1 \setminus T_2$ contains at lest n elements and has a zero-subsequence T_3 of length t_3, $t_3 \leqslant t_2$ by the maximality of T_2. But $t_2 \leqslant [n/2]$, whence $t_3+t_2 \leqslant n$. This contradicts the maximality of T_2.

1.11. Answer: $\frac{1}{p}(C_{2p}^{p}-2)+2$. This problem is from IMO1995. We take a solution in [1]. We may assume that the sequence is a union of two identical sets B and C, where $B = C = \{0,1,2,\ldots,p-1\}$.

Split all the p-element subsequences of our sequence except B and C into the groups of p subsequences. For every set $X = \{x_1,\ldots,x_k\} \subset B$ denote by $X+\ell$ the set $\{x_1+\ell,\ldots,x_k+\ell\} \subset B$, where addition is mod p (i.e. $X+\ell$ is a usual cyclic shift of the set X) For a sequence A we place into one group the following subsequences

$$A_0 = A;\ A_1 = \big((A\cap B)+1\big)\cup(A\cap C);\ A_2 = \big((A\cap B)+2\big)\cup(A\cap C);\ \ldots, A_{p-1} = \big((A\cap B)+p-1\big)\cup(A\cap C)\,.$$

If $|A\cap B| = q$, $0 < q < p$, then the sums of subsequences A_i and A_j differ by $(j-i)q$. Therefore all the sums of subsequences in one group are pairwise distinct. Hence the sum of exactly one subsequence is divisible by p.

Thus we have $\binom{2p}{p}$ p-element sequences, two of them (with zero sum) we throw off and between others $1/p$ part has a zero sum. So the answer is $\frac{1}{p}(C_{2p}^{p}-2)+2$.

1.12. Let $A = (a_1,\ldots,a_n)$ be a given sequence. In the solution of problem 1.1 we describe how to construct a zero-subsequence elements of which run successively in a given numbering. I.e. the sequence we have obtained in such a way has a form a_i, a_{i+1}, …, a_j, where $1 \leqslant i < j \leqslant n$. The sequences of this type we will be referred to as intervals. After we have constructed several zero-sequences we can try to “mix” all numbers by changing enumeration in order to each of our sequences would not be an interval. In case of success of these attempts we can construct one more zero-subsequence by the method of the problem 1.1. This subsequence will be an interval and therefore it will not concide with those that we have constructed previously. Proceeding in this manner we will construct $n-1$ desired subsequences.

In order that this plan works, it remains to prove the following statement.

Lemma. Let $k \leqslant n-2$ subsequences (of the initial sequence A) be given such that they have length n and contain from 2 to $n-1$ elements each. Then we can rearrange the elemnts of A in such a manner that none of subsequences will be an interval in a new enumeration.

Proof. When we do not look after the order of elements we will use the word "set" instead of "sequence".

We use induction by n. Base of induction $n \leqslant 4$ can be easily checked by consideration of all possibilities. Suppose that the statement is proven for some $n \geqslant 4$. Then we will show that it is valid for $n+1$ by proving the following statement.

Let the sequence $B = (b_1, b_2, \dots, b_{n+1})$ be given. Let $D_1, \dots, D_k$ be its subsequences containing from 2 to n elements each, $k \leqslant n-1$. Then there exists a new enumeration of the sequence B such that all sequences D_i are not intervals.

Since we have at most $n-1$ sets D_i, then there exists an element b_k that belongs to at most one two-element set D_i; without loss of generality we may assume that this is b_{n+1}. Remove the element b_{n+1} from the initial sequence and from all subsets containing it. If a pair D_i containing b_{n+1} exists then drop it. If a set $D_j = \{b_1, \dots, b_n\}$ exists then drop it too. If neither first nor second set exists then drop an arbitrary set D_m.

We can apply induction hypothesis to the remaining elements and sets because now all the sets contain from 2 to $n-1$ elements each. We try to find a place in the obtained sequence where we can insert an element b_{n+1} in order to satisfy all conditions. If an arbitrary D_m has been dropped it is sufficient to insert b_{n+1} in such a manner that elements of D_m would not run successively. Clearly it is possible. Otherwise if we have had a pair D_i then we can not insert b_{n+1} on the two positions adjacent to the second element of the pair. And finally if we have had the set D_j then now the "prohibited" positions are at the beginning and at the end of the sequence. Since total number of positions for b_{n+1} is $n+1 \geqslant 5$, we have at least one "allowed" position. Insertion of b_{n+1} in this position provides us a desired sequence.

2.1. The elements of $\mathbb{Z}_2^d$ are all possible collections of d zeros and ones equipped by the operation of addition modulo 2 in each coordinate. There are 2^d such collections, and the sum of any two different collections is not a zero collection. If we take 2^d+1 collections, then two of them are identical. Their sum will be equal to zero.

2.2. The upper estimate follows from the fact that if a sequence consists of $(n-1)n^d+1$ elements of $\mathbb{Z}_n^d$ then some element occurs at least n times. For the lower bound let us give an example. We consider 2^n collections of the form $(m_1, \dots, m_d)$, where each m_i takes value 0 or 1. We construct a sequence containing each of these collections exactly $n-1$ times. It is clear that the sum of any n elements of this sequence is not equal to 0.

2.3. Suppose a sequence containing $s(n_1,d) + n_1(s(n_2,d)-1)$ elements is given. We will consequtively choose groups, containing n_1 elements, with the sum in each group divisible by n_1, using the definition of $s(n_1,d)$. We can choose $s(n_2,d)$ groups in this way. Now divide the sum of elements in each group by n_1 thus obtaining a sequence of $s(n_2,d)$ elements. After this we can choose n_2 elements of this sequence with the sum divisible by n_2.

2.4. Here we give an example of nine elements of $\mathbb{Z}_3^3$, such that no three of them have zero sum. It was given in [10]:

$$\begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}.$$

Furthermore, if we take each vector two times, we get an example of the sequence of 18 elements, such that no three of them have zero sum.

2.5. See [8]. For odd n we can take the following subset of $\mathbb{Z}_n^2$:

$$(0,0),(0,1),\dots,(0,n-2),\ (1,1),(1,2),\dots,(1,n-1).$$

This set conyains $2n-2$ elements. If its subset contains n elements then it has nonzero sum. For even n we can similarly consider the following subset of $\mathbb{Z}_n^2$:

$$(0,0),(0,1),\dots,(0,n-1),\ (1,0),(1,1),\dots,(1,n-1).$$

This set conyains $2n$ elements. If its subset contains n elements then it has nonzero sum.

2.6. This fact is taken in [8] but the proof there is too technical. If we consider integer vectors a_1, a_2, a_3, a_4 then $\frac{a_1+a_2+a_3+a_4}{4} = \frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2}$. Split 9 marked points into 4 groups according to the parity of both coordinates. In each group $x_1, x_2, \dots, x_k$ compute $\frac{x_1+x_2}{2}, \frac{x_1+x_3}{2}, \dots, \frac{x_1+x_k}{2}$. It yields $k-1$ integer vectors which are pairwise distinct as elements of $\mathbb{Z}_2^2$. Having considered all 4 groups, we get at least $9-4=5$ vectors with integer coordinates.

We can choose two vectors with the coinciding parities of both coordinates of them. The pairs of initial vectors corresponding to them were received from different groups, so they provide the required 4-tuple.

2.7.

2.8. This statement has been proven in AlonDubiner2 by non elementary methods.

3.1. The simple case $|A|+|B|>p$ is left to the reader. We may therefore consider subsets satisfying the condition $|A|+|B|\leqslant p$.

Assume that, in contradiction with our assumptions, the set A and B satsfy the inequality

$$|A+B|<|A|+|B|-1\,. \tag{$*$}$$

For definiteness, assume that $|B|\geqslant|A|$. Among all pairs satisfying the above inequality $(*)$ we choose pairs such that $|A+B|$ achieves its minimum possible value. Among those pairs take an arbitrary pair A, B such that the cardinality of the smaller A achieves its minimum.

If $|A|=1$ then the inequality is obvious.

Consider the case $|A|\geqslant 2$. If instead of the set A we consider for an arbitrary $c\in\mathbb{Z}_p$ the shifted set $A+c$, then the set $A+B$ is also shifted by c, and in particular its number of elements remains the same.

Assume that we may shift the set A in such a way that the sets A and B have a nonempty intersection, but $A\not\subset B$. Consider the sets $A_1=A\cap B$, $B_1=A\cup B$. By definition $|A_1|+|B_1|=|A|+|B|$. But $A_1+B_1\subset A+B$, whence $|A_1+B_1|\leqslant|A+B|$ and the pair A_1, B_1 satisfies the inequalty $(*)$. Since $|A_1|<|A|$, the set A is not minimal, which contradicts our assumptions.

Now we prove that we may shift the set A in such a way that the sets A and B have a nonempty intersection but $A\not\subset B$. Indeed, assume that such a shift is impossible, i.e. any shift of A is either contained in B or does not intersect B. Let $a,\ b\in A\cap B$, $A\subset B$. Shift the set A by $b-a$ and denote $A'=A+(b-a)$ (all arithmetical operations in our formulas should be understood mod p). The point a is taken by our shift to the point b, and therefore A' and B have a nonempty intersection, whence $A'\subset B$ and $b,\ b+(b-a)\in A'\subset B$. We now shift A' by $b-a$ and obtain the set A''. Observe that A'' and B have a nonempty intersection. Proceeding inductively we obtain that all points

$$a,\ b,\ b+(b-a),\ b+2(b-a),\ b+3(b-a),\ \ldots$$

belong to the set B. But the set of all such points exhausts the whole set $\mathbb{Z}_p$, and we arrive at a contradiction.

3.2. First assume that we have proved our statement for prime values of n and derive it for all n by induction on the number of prime divisors of n. Let $n=pm$, where p is prime. Let $a_1,\ a_2,\ \ldots,\ a_{2n-1}$ be our sequence.

By our statement for prime p we may choose an array I_1 of p elements with zero sum mod p. From the remaining $2(m-1)p-1$ elements we may choose another such array I_2. Proceeding in this way we obtain $2m-1$ distinct arrays $I_1,\ I_2,\ \ldots,\ I_{2m-1}$. (Note that after $2m-2$ arrays have been chosen the remaining number of elemnts is at least $2pm-1-(2m-2)p=2p-1$.) For each i, $1\leqslant i\leqslant 2m-1$, denote $a'_i=\frac{1}{p}\sum_{j\in I_i}a_j$. By the inductive hypothesys the sequence a'_i contains a zero sum sequence of m elements. We denote this elements by $a'_{i_1},\ a'_{i_2},\ \ldots,$ a'_{i_m}. The union of arrays $I_{i_1},\ \ldots I_{i_m}$ is the desired set of pm elements with sum divisible by n.

We now prove our statement for prime p. Arrange our sequence in increasing order: $0\leqslant a_1\leqslant a_2\leqslant\ldots\leqslant a_{2p-1}$. If $a_i=a_{i+p-1}$ for some i, then $a_i+a_{i+1}+\ldots+a_{i+p-1}=pa_i=0$ (in $\mathbb{Z}_p$) and the statement is proved. In the opposite case denote $A_i=\{a_i,a_{i+p-1}\}$ for $1\leqslant i\leqslant p-1$. Using the Cauchy$-$Davenport theorem several times we obtain $|A_1+A_2+\ldots+A_{p-1}|=p$ (one may also use problem 3.3 here). Therefore, every element of $\mathbb{Z}_p$ may be represented as a sum of $p-1$ elements of the sequence $a_1,\ a_2,\ \ldots,\ a_{2p-2}$. But then the element $-a_{2p-1}$ also admits such a representation, and we obtain the desired zero subsequence.

3.3. An obvious induction.

3.4. This problem is taken from [8].

The statement is obvious if some element occurs p times in S. Assume that each element occurs at most $p-1$ times. Denote $A_1=\{a_1,\ldots,a_s\}$, and split other elements of S into $p-1$ sets $A_2,\ \ldots,\ A_{p-1}$. By the Cauchy–Davenport theorem,

$$|A_1+\ldots+A_p|\geqslant\min\{p,\sum|A_i|-p+1\}=\min\{p,2p-1-p+1\}=p\,.$$

Hence $A_1+\ldots+A_p=\mathbb{Z}_p$, which implies the desired statement.

3.5. A composition of two axial symmetries is a rotation. We can interpret a rotation of a 12-gon as an element of the group $\mathbb{Z}_{12}$. Let us decompose our collection of transformations into pairs of two symmetries or two rotations (there will be one transformation without a pair; we will not use it). Fix an arbitrary order of elements in each pair. There are 23 pairs, and the composition of elements in each pair is a rotation; these compositions form a collection of 23 elements of $\mathbb{Z}_{12}$. Applying the Erdős – Ginzburg – Ziv theorem to this collection, we obtain the required result.

3.6. Let us split all the elements of T into h sets $A_1, A_2, \ldots, A_h$ in the following *canonical* way: we denote by A_i the set of all elements appearing in T at least i times. Then any element has equal multiplicities in T and in the sets $A_1, \ldots, A_h$; moreover, A_1 contains all different elements of T. All the sums of not more than h elements of T form the set $A_1 + \sum_{i=2}^{h} \big(A_i \cup \{0\}\big)$. By the Cauchy–Davenport theorem, we obtain

$$\Big|A_1 + \sum_{i=2}^{h} \big(A_i \cup \{0\}\big)\Big| \geqslant \min\big\{p,\ |A_1| + |A_2 \cup \{0\}| + \ldots + |A_h \cup \{0\}| - h + 1\big\} = p$$

as required.

3.7. Using the Erdős – Ginzburg – Ziv theorem one can find a zero subsequence of k numbers. Delete them from the sequence and repeat the procedure. This process can be repeated until there are less than $2k - 1$ numbers left in the sequence. When this is the case, we obtain m/k groups with zero sums as required.

3.8. This statement is proved in [13].

Assume the contrary. As in the proof of the Cauchy–Davenport theorem, we may restrict ourselves to the case $|A| + |B| - 1 \leqslant n^d$. From all pairs (A, B) contradicting the statement of the problem, choose a pair with minimal $|A|$. Note that there exists an element $b^* \in B$ such that $b^* + A \not\subset B$. Actually, assuming the converse we have the equality $a_1 + B = B$ for any $0 \neq a_1 \in A$. In particular, $a_1 + b_i = 0 \in B$ for some $b_i \in B$ which is impossible.

We will use this element to redistribute our pair of sets. Denote by A^* the set of all $a \in A$ for which $a + b^* \notin B$. Let $A' = A \setminus A^*$, $B' = B \cup (b^* + A^*)$. Obviously, $0 \notin A^*$, hence $0 \in A'$. Moreover $0 \in B' \supset B$.

The equation $a' + b' = 0$ has only a trivial solution on the sets A', B'. Actually, if $a_1 + (b^* + a_2) = 0$, where $a_1 \in A'$, $a_2 \in A^*$, then $(a_1 + b^*) + a_2 = 0$, which is impossible since $a_1 + b^* \in B$, $a_2 \in A$, $a_2 \neq 0$. On the other side, if $a_1 + b_1 = 0$ for $0 \neq a_1 \in A'$, $b_1 \in B$, then we obtain a nontrivial solution of the initial equation again.

Additionally, recall that $A' + B' \subset A + B$.

Finally, we obtain a new pair (A', B') satisfying the same conditions though with $|A'| < |A|$. Contradiction with minimality of $|A|$.

The solution is valid for any finite abelian group.

3.9. This statement is taken from [11]. The Erdős – Ginzburg – Ziv theorem implies that $r \leqslant k - 2$.

If we add some number to all elements (this operation will be referred to as *a shift* of a sequence), then all k-sums remain the same, so one can consider only the case when 0 has the maximal multiplicity in A. Denote by L a subsequence of A, consisting of all zeroes. Obviously, $\ell \leqslant k - 1$. Consider all subsequences in $A \setminus L$ of length not exceeding $k - 1$. Let S be the subsequence of maximal length in this set. Denote $s = |S|$, $\ell = |L|$ (possibly $S = \varnothing$ and $s = 0$). Then $\ell + s \leqslant k - 1$ (conversely, we obtain a zero subsequence by adding to S some zeroes from L).

Then, $|A \setminus (L \cup S)| \geqslant r + 1$. Let T be an arbitrary subsequence of length r in $A \setminus (L \cup S)$. Let h be the maximal multiplicity of an element in T. Then $h \leqslant \ell$ by the definition of ℓ. Split T into h sets $X_1, \ldots, X_h$ in a canonical way and denote $X_i' = X_i \cup \{0\}$ $(i = 1, \ldots, h)$. Note that $0 \notin T$ and for any $1 < j \leqslant h$ no j elements of T have zero sum. On the contrary, we can add these elements to S obtaining a new zero subsequence of length $j + s \leqslant h + s \leqslant \ell + s \leqslant k - 1$, which contradicts to the choice of S. Hence, we have verified the conditions of the Kemperman–Scherk theorem. Applying this theorem, we obtain

$$|X_1' + \ldots + X_h'| \geqslant |X_1| + \ldots + |X_h| + 1 = r + 1\,.$$

In other words, if we add h zeroes from L into T, then the obtained sequence has at least $r + 1$ h-sums. Adding the remaining $(k + r) - (r + h)$ elements of A to all these sums, we obtain $r + 1$ different k-sums as required.

The Erdős – Ginzburg – Ziv theorem is in fact a corollary of this statement. The arguments are valid for any finite abelian group.

3.10. This result is a consequence of the Kemperman–Scherk theorem. Actually, there is nothing to prove if $0 \in S$. Suppose $0 \notin S$. Distribute elements of S into h subsets $B_1, \ldots, B_h$ in a canonical way and let $A_i = \{0\} \cup B_i$ for $1 \leqslant i \leqslant h - 1$. If an equation $a_1 + a_2 + \ldots + a_{h-1} = 0$ has a nontrivial solution with $a_i \in A_i$, then we obtain the required result. In the opposite case we have

$$|A_1 + A_2 + \ldots + A_{h-1}| \geqslant \Big(\sum_{i=1}^{h-1} |A_i|\Big) - (h - 1) + 1 = (n^d + h - 1 - |B_h|) - (h - 1) + 1 = n^d + 1 - |B_h|.$$

This implies that there exists an element of $\mathbb{Z}_n^d$ of the form $(-b)$, where $b \in B_h$, which can be represented as a sum of the form $a_1 + a_2 + \ldots + a_{h-1}$, $a_i \in A_i$. Then $a_1, a_2, \ldots, a_{h-1}, b$ is a required zero subsequence.

3.11. This statement is found in [8].

The p-sums remain the same under a shift. Hence, having made a shift we may assume that 0 has the maximal multiplicity h. Let T be a subsequence consisting of all nonzero elements, and s be the sum of its elements. Assume that the statement is not valid and $h \leqslant p - k$. Then $|T| \geqslant p$. Since $0 \notin T$, s can be represented as the sum of not more than h elements of T by problem 3.6. Let Q be a subsequence formed by these elements, and set $T_1 = T \setminus Q$. Clearly, T_1 is the zero sequence, and $p - h \leqslant |T_1| \leqslant |T| - 1$.

Suppose $|T_1| < p$. Then one can easily obtain a zero sequence by adding some zeroes to T_1. Hence the only case remaining is $|T_1| > p$. Again, applying problem 3.6, we can eliminate not more than h elements from T_1, obtaining the subsequence T_2 with zero sum, and so on. Finally, for some T_i we will obtain $p - h \leqslant |T_i| \leqslant p$, QED.

3.12. This statement is equivalent to the problem 3.10.

3.13. Found in [5].

Assume the contrary: each element has multiplicity not more than $2k - n$ in S. Note that for $a, b \in S$, $a \neq b$, elements a, b and $a + b$ are pairwise distinct since S does not contain zero subsequences.

Lemma. Suppose $a, b, c \in S$ are pairwise distinct elements. Then there are at least 6 different sums modulo n among all the sums of the set $\{a, b, c\}$.

Proof. Consider the following equations in $\mathbb{Z}_n$:

$$a + b = c, \quad a + c = b, \quad b + c = a\,.$$

Suppose that two of them are satisfied (wlog 1st and 2nd). Adding we obtain $2a = 0$ which is impossible since n is odd. If, on the contrary, two of these equalities (again 1st and 2nd) are NOT satisfied, then one can easily check that the sums a, b, c, $a+b$, $a+c$, $a+b+c$ are pairwise distinct. Actually, if, for instance, $a+b+c = a$ then $b+c$ is a zero sum. The lemma is proved.

Let us choose one by one nonintersecting triples of pairwise distinct elements (and eliminate them from the sequence). Assume that we can choose j triples $A_1, A_2, \ldots, A_j$, and the rest of the sequence does not contain such a triple. Then the rest contains only 2 different elements a and b with multiplicities λ and μ, $\lambda \geqslant \mu \geqslant 0$. Our assumption implies that $\lambda \leqslant 2k - n$. We can make μ pairs $\{a, b\} = A_{j+1} = \cdots = A_{j+\mu}$ and $\lambda - \mu$ singles $\{a\} = A_{j+\mu+1} = \cdots = A_{j+\lambda}$ from these elements.

We have split our sequence into $j + \lambda$ sets A_i. So,

$$k = 3j + \lambda + \mu\,.$$

For any A_i we introduce the set ΣA_i of all sums of subsets of A_i. By Lemma we have $|\Sigma A_i| \geqslant 6$ for triples; moreover, $|\Sigma A_i| = 3$ for all pairs and $|\Sigma A_i| = 1$ for singles. It follows that

$$\sum_{i=1}^{j+\lambda} |\Sigma A_i| = 6j + 3\mu + (\lambda - \mu) = 2(3j + \mu + \lambda) - \lambda = 2k - \lambda \geqslant n\,.$$

It remains to apply the statement of the problem 3.12.

Remark. The statement of the problem is also valid for even values of n. For this case, one should add some technical corrections into the solution; these corrections will deal with the case when $n/2$ is present in S.

3.14. A straightforward consequence of problem 3.12 where all sets are supposed to be equal.

3.15. Suppose a sequence of $2n - 1$ elements of $\mathbb{Z}_n$ is given in such a way that it has a unique zero subsequence of length n. Let us remove any element appearing in this zero subsequence. Then the remaining sequence S does not have a zero subsequence of length n at all. Let a be an element with maximal multiplicity in S, and s be its multiplicity. Again, we will consider the shifted sequence $T = S - a$ instead of S, 0 being its member with multiplicity s. Let T_1 be the subsequence consisting of all nonzero elements of T, and S_0 be the subsequence consisting of all zeroes.

Clearly, $s < n$. If $s = n - 1$, then $|T_1| = n - 1$ and T_1 does not contain nonempty zero subsequences at all. By problem 1.2, T_1 consists of $n - 1$ equal elements. Hence S has only two different elements, each being with multiplicities $n - 1$.

Suppose $s \leqslant n - 2$. Then $|T_1| \geqslant n$, and the multiplicities of its elements do not exceed s. By problem 3.10, we may choose a zero subsequence S_1 in T_1 of length $s_1 \leqslant s$. If $s + s_1 \geqslant n$, then we may construct a zero sequence of length n, adding some zeroes to S_1. If $s+s_1 \leqslant n-2$, then we may find another zero subsequence $S_2 \subset T_2 = T_1 \setminus S_1$ of length $s_2 \leqslant s$, and so on. Finally, we will obtain either $s+s_1+\dots+s_k \geqslant n$ (hence we may find a zero subsequence of length n by adding some zeroes to $S_1 \cup \dots \cup S_k$) or $s+s_1+\dots+s_k = n-1$.

If $T_{k+1} = T_1 \setminus (S_1 \cup \dots \cup S_k)$ does not contain zero subsequences, then by problem 1.2 all its elements are equal, which is impossible since the maximal multiplicity is $s < n-1$. Otherwise there exists a zero subsequence U of length u, and $u+s+s_1+\dots+s_k \geqslant n$. Removing from the sequence $S_0 \cup S_1 \cup \dots S_k \cup U$ a suitable number of S_i and then a suitable number of zeroes, we will obtain a zero subsequence of length n. Contradiction.

So, we have shown that S has only two different elements, each being with multiplicities $n-1$. Now let us return the removed element and throw another one appearing in the zero subsequence. Again, we will obtain a sequence with exactly two values; this means that the first deleted element was equal to some other. It follows that the sequence has the required form.

3.16. This is a particular case of another theorem of Erdős–Eggleton [7].

Let $f(k)$ be the least possible number of elements of $\mathbb{Z}_n^d$ which may be represented as a sum of some subsequence of S, where S is an arbitrary k-sequence without zero subsequences. Let us prove that $f(k) \geqslant 2k-1$ by induction on k. Obviously, $f(1) = 1$.

Suppose that $f(k) \geqslant 2k-1$. Consider a sequence $S = (a_1, a_2, \dots, a_{k+1})$ without zero subsequences. We should check that there are at least $2k+1$ elements which can be represented as a sum of some its elements.

Case 1. Some element of S (say a_{k+1}) cannot be represented as a sum of some other elements of S. By the induction hypothesis, the set of all sums of $S \setminus \{a_{k+1}\}$ consists of at least $2k-1$ elements, and this set contains neither a_{k+1} nor $\sum_{i=1}^{k+1} a_i$ (otherwise the difference of this sum and its representation yields a zero subsequence). Moreover, these elements are obviously different. Hence we have found $2k+1$ different sums.

Case 2. Each element of S can be represented as a sum of some other elements. Recall that S does not contain zero subsequences, hence we can apply the Kemperman–Sherk theorem to the sets $A = B = \{0, a_1, \dots, a_{k+1}\}$ obtaining $|A+B| \geqslant 2k+3$. Moreover, each element of the form $2a_i$ can be represented as the sum of different elements of S since one can change a_i by its representation as a sum of some other elements. Hence, each nonzero element of $A+B$ can be represented as the sum of some elements of S, hence we have in fact found $2k+2$ of such elements.

3.17. A result from [8]. Since we deal with p-sums, we may shift our sequence again. Then we can assume that there is exactly one 0 and exactly two entries of any other element of $\mathbb{Z}_p$ in our sequence. Then

$$(a_n) = ((0,z),\ (1,x_1),(1,y_1),\ (2,x_2),(2,y_2),\ \dots,\ (p-1,x_{p-1}),(p-1,y_{p-1}))\,,$$

where $x_i \neq y_i$ for all i (since (i,x_i) and (i,y_i) are different elements of M). Note that $0+1+2+\ldots+(p-1) = 0$ in $\mathbb{Z}_p$. Denote $C_i = \{x_i, y_i\}$. By the Cauchy–Davenport theorem 3.3, we have

$$|C_1 + C_2 + \ldots + C_{p-1}| \geqslant 2(p-1) - (p-1) + 1 = p\,.$$

This means that zero may be represented as $0 = \sum_{i=0}^{p-1} z_i$, where $z_0 = z$ and for each i either $z_i = x_i$ or $z_i = y_i$. The corresponding elements of our sequence form a required subsequence.

3.18. This is a result of Olson [14].

Let us show that the number in this statement cannot be made smaller. Take d "basic" vectors

$$e_i = (0,0,\dots,0,1,0,\dots,0)$$

(the unit is on the ith place in the vector e_i), each having multiplicity $p-1$.

The proof of the remaining part of the statement is much more complicated.

3.19. This fact is proved algebraically in [4].

3.20.

3.21.

References

[1] Школьные олимпиады. Международные математические олимпиады / Сост. А. А. Фомин, Г. М. Кузнецова. М: Дрофа, 1998.

[2] Alon N. Subset sums // J. Number Theory. 1987. Vol. 27. P. 196–205.

[3] Alon N., Dubiner M. A lattice point problem and additive number theory // Combinatorica. 1995. Vol. 15. P. 301–309.

[4] Alon N., Dubiner M. Zero-sum sets of prescribed size. In: Combinatorcs, Paul Erdős is Eighty. Bolyai Society, Math. Studies, Keszthely. Hungary, 1993. P. 33–50.

[5] Bovey J. D., Erdős P., Niven I. Conditions for a zero sum modulo n // Canad. Math. Bull. 1975. Vol. 18. № 1. P. 27–29.

[6] Davenport H. On the addition of residue classes // J. London Math. Soc. 1935. Vol. 10. P. 30–32.

[7] Eggleton R. B., Erdős P. Two combinatorial problems in group theory // Acta Arithmetica. 1972. Vol 19. P. 111-116.

[8] Gao W.D., Thangadurai R. A variant of Kemnitz conjecture // J. of Comb. Theory. Ser. A. 2004. Vol. 107. P. 69–86.

[9] Halberstam H., Roth K.F., Sequences. New York, Heidelberg, Berlin: Springer-Verlag, 1966.

[10] Harborth H. Ein Extremalproblem für Gitterpunkte // J. Reine Angew. Math. 1973. Vol. 262/263. P. 356-360.

[11] Hong Bing Yu. A simple proof of a theorem of Bollobás and Leader // Proc. of the AMS. 2002. Vol. 131. № 9. P. 2639–2640.

[12] Kemnitz A. Exremalprobleme für Gitterpunkte. Ph.D.Thesis. Technische Universität Braunschweig, 1982.

[13] Moser L., Scherk P. Distinct elements in a set of sums // Amer. Math. Monthly. 1955. Vol. 62. P. 46–47.

[14] Olson J.E., A combinatorial problem on finite Abelian groups, II // J. Number Theory. 1969. Vol. 1. P. 195–199.

[15] Olson J.E., An addition theorem modulo p // J. Comb. Theory. 1968. Vol. 5. P. 42–52.

[16] Thangadurai R. Non-canonical extensions of Erdős–Ginzburg–Ziv theorem // Integers: Electronic J. of Comb. Number Theory. 2002. V. 2.