Groups and Mosaics

Mistakes

It is reasonable to fix some mistakes in the problems text.

♦ **A4.** *Let us consider a plane divided into* `convex` *7-gons which diameters are less or equal to* $\mathbf{1}$. *Fix a point* O. *Let* $N(R)$ *be the number of 7-gons falling into the circle with diameter* R *and center* O. *Prove that there exists* $\lambda > 1$ *such that* $N(R) > \lambda^R$.

♦ **B1.** *Let* $ba = ab$ *be a defining relation. Prove that any word can be transformed to the form* $a^m b^n$*; here* m, n *are integers.*

Additional problems

♦ **A5.** *Can one cut a plane to convex 7-gons such that diameters of the 7-gons are less or equal to 1 and any unite circle intersects with less then million of them?*

♦ **B7.5a.** *Consider a group over an alphabet* $\{a, b\}$. *Prove the following statement: if for any* x *from a group* $x^3 = 1$, *then the group is finite.*

♦ **B7.5b.** *Consider a group over an alphabet* $\{a, b, c\}$. *Prove the following statement: if for any* x *from a group* $x^3 = 1$, *then the group is finite.*

Some additional technics

Now we shall give the following notation. Consider a cutting of a plane to some polygons (cells). This cutting is called a *map* and is denoted by U. An oriented edge of a cutting U is called an *edge of a map.* So if there exist an edge e, then there exist an opposite oriented edge e^{-1}. This edge e^{-1} belongs the same points of the plane as the e one. Let us make an agreement that outlines of all cells should be read по часовой стрелке. Consider the outline of all map. This outline belongs n edges. Let us denote these edges e_1, e_2, $\ldots e_n$ using the orientation of the map. A lap $e_1 \ldots e_n$ is called an *outline of a map* U. We can define the outline of a cell by the same way. We shall consider an outline of a map regardless of cyclic shift. Suppose an outline (of a cell, or of a map) $e_1 \ldots e_n$ belongs an edge e. A chain of the edges e_1, e_2, $\ldots e_n$ is called a *way* if the end of e_i is congruent to the begin of e_{i+1} for any $i = 1, \ldots, n-1$.

A subpath is similar to a subword: p is a *subway* of g if $q = p_1 p_2$ holds for some ways p_1 and p_2. Consider a finite alphabet L. We denote $\bar{L} = L \cup L^{-1} \cup 1$, here L^{-1} is an alphabet of inverse letters.

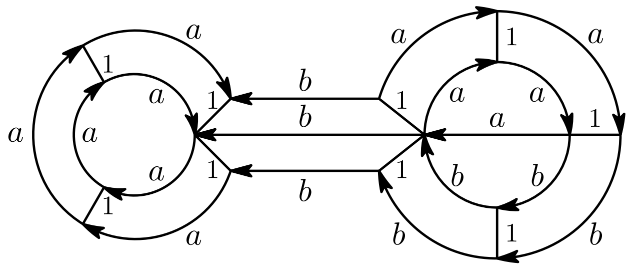
Let us assign a letter $\phi(e)$ from $\bar{L}$ to each edge e of map U. The map U is called a *diagram over* U if $\phi(e^{-1}) = \phi(e)^{-1}$. Consider a path $p = e_1 \ldots e_n$ in the diagram U over L. A *label* $\phi(p)$ is a word $\phi(e_1) \ldots \phi(e_n)$ in the alphabet $\bar{L}$. By definition, put $\phi(p) = 1$ if

$n = |p| = 0$. It is easy to see that a label of cell (or diagram) outline is defined up to cyclical shift. So it is a cyclical word.

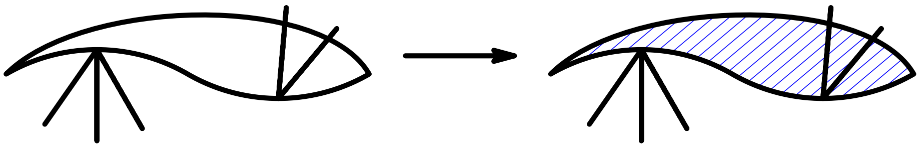
A cell K is called a R-*cell* if it's outline label $\phi(p)$ is graphically equal (up to cyclical shift) to some word from defining relations or its inverse (up to pasting some amount of symbols $\mathbf{1}$). It is clear that if we choose the beginning and direction of "reading" and ignore the symbol $\mathbf{1}$, then we can read a defining relation word.

A cell K is called a *0-cell* if it's outline label $e_1 \dots e_n$ is graphically equal $\phi(e_1) \dots \phi(e_n)$, where all $\phi(e_i) = \mathbf{1}$ (= means a graphical equality) OR if for some $i \neq j$ $\phi(e_i) = a$, $\phi(e_j) = a^{-1}$ a belongs to alphabet and for all other $k \neq i, j$ $\phi(e_k) = \mathbf{1}$. An edge is called a $\mathbf{0}$-*edge* if it's label is equal to $\mathbf{1}$. An edge is called a U-*edge* if it's label is non trivial word. *A length* $|p|$ of an arbitrary path is a number of it's U-edges. *Perimeter* of a cell or a diagram is just a length of it's outline.

There are no 0-cells in the examples presented above. However, it is reasonable to introduce them by following reason. The examples shown on the figures 1–3 are truly disk diagrams: if we delete the outline, then the diagram doesn't brake onto several parts. Diagram on the picture 4 is not a disk: if we delete the outline, then it divides into several parts. This problem cause some technical difficulties. For example, we need to cut the subdiagram for induction reason. If we use 0-cells, then we can make a disk diagram with the diagram on the picture 4. It is reasonable to think of 0-cells as "thin" cells (or "fat" edges) and 0-edges as extremely "short" edges.



pic. 5



pic. 6

Thus, a diagram over an alphabet U is called a *diagram over group* G defined by the relations $R_1 \dots R_n$ if it's every cell is R-cell or $\mathbf{0}$-cell.

It is reasonable to do a $\mathbf{0}$-*cutting* of a diagram. Let a cell be a polygon. We draw a second polygon inside the first one. The second polygon is similar to the first one. Let us connect corresponding vertexes of the polygons with additional edges. Then we label the second polygon with letters as the first one. The additional edges are $\mathbf{0}$-edges. Let us label them with $\mathbf{1}$'s. So we obtain the $\mathbf{0}$-cutting of a cell. Similarly, we can obtain a *doubling* of a path. Draw an additional edge for every edge of a path. Then we label the additional edges by the same way as the corresponding original edges. So we obtain two paths with the same beginning and ending.