## FUNCTIONAL EQUATIONS

**Second stage**

The improvement for First Stage:

**9b.** Find all continuous solutions for the Cauchy equation

$$f(x+y) = f(x)f(y) \quad (x, y \in R).$$

F) Suppose a function $g$ satisfies the equation

$$g(x+y) = g(x) + g(y)$$

where the tuple $(x, y)$ belongs to some subset $Z$ of the plane $R^2$. If $g$ can be extended to a function $f$ satisfying the same equation for all $x, y \in R^2$ then we say that $f$ is an *additive extension* of $g$.

**13.** Show that if $Z$ is the unit square then any function $g$ satisfying the Cauchy equation on $Z$ has a unique additive extension to the whole plane.

**14**. Give an example of an infinite set $Z$ and a function $g$ which satisfies the Cauchy equation on $Z$ but has no additive extension from $Z$ to $R^2$.

The above results have an application, for instance, in the following economic-mathematical model described in [1], pp. 95-96. Suppose we have to divide an amount $S$ of money between $m > 2$ competing projects. Each of $n$ experts makes a recommendation (expert $j$ suggests to grant the project $i$ with the sum $\xi_{ij}$), and finally the 'consensus' allocation is given by some function

$$\phi_i(\xi_{i1}, \ldots, \xi_{in}).$$

Observe that for each project the consensus allocation is determined by the sums recommended by the experts for this project only, but the form of this dependence may vary for different projects. We impose two natural requirements.

(i) If all experts allocate zero sum to some project then this project obtains 0 in the consensus allocation:

$$\phi_i(0, \ldots, 0) = 0 \quad (i = 1, \ldots, n).$$

b) If all the allocations recommended by the experts exhaust the sum $S$ then this is true for the consensus allocation as well: $\sum\limits_{i=1}^{m} x_{ij} = S$ $(j = 1, \ldots, n)$ implies $\sum\limits_{i=1}^{m} \phi_i(x_{i1}, \ldots, x_{in}) = S$.

Show that under the above conditions, all the functions $\phi_i$ have the same (not depending on $i$) form $\sum \omega_j \xi_j$ where $\omega_j \geq 0$, $\sum \omega_j = 1$.

G) **16.** Find all continuous real functions of a positive real variable which satisfy the equation

$$f(xy) = a(x) + b(x)c(y).$$

Many of you know that the integral of a power function is again a power function (with a coefficient) with the only exception: the integral of $1/x$ is the logarithm (all integrals are, of course, defined up to an additive constant). In a standard course of calculus, this fact is proved with the help of differentiation, and the cases of degree $-1$ and of all other degrees are treated separately.

**17.** Find the integral of $x^a$ where $x$ is a positive real variable, $a$ an arbitrary constant, using the results of Problems 16, 9a and 9c as the base for your argument. It is not allowed to differentiate!

H) **18.** Find all continuous solutions of the **d'Alembert equation**

$$f(\phi+\psi)+f(\phi-\psi)=2f(\phi)f(\psi)$$

under the condition $f(\pi/4)=\sqrt{2}/2$.

**19.** Now can you present a functional equation defining

(a) the sine function $\sin\ x$?

(b) the tangent function $\tan\ x$?

$*$**20.** Using results of Problems 8 and 18, show that the vector addition in 3-dimensional Eucledean space is the only operation on pairs of such vectors which satisfies the following conditions:

(i) if both vectors are subject to the same rotation then the result of the operation also is subject to the same rotation;

(ii) the operation is commutative and associative;

(iii) two vectors pointing in the same direction yield a vector of the same direction whose length is the sum of the lengths of our initial vectors;

(iv) the sum of two vectors of equal length depends continuously on their angle.

**Bibliography**

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