HILBERT'S 13-TH PROBLEM AND BASIC PLANAR SETS

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PROBLEMS PROPOSED BEFORE THE INTERMEDIATE FINISH.

Motivation.

In the course of the solution of the Hilbert's 13-th problem the notion of *basic embedding* appeared. The main result of the present sequence of problems (problem 8b) is an elementary solution of 'half' of the Arnold problem on the characterization of basic subsets of the plane. The most important unsolved problems here concern the characterization of *smoothly basic* subsets of the plane.

The more difficult problems are marked by a star, and unsolved problems by two stars. If the statement of a problem is an assertion, then it is required to prove this assertion.

Discontinuously basic subsets.

1. (a) Is it true that for any four numbers $f_{11}, f_{12}, f_{21}, f_{22}$ there exist four numbers g_1, g_2, h_1, h_2 such that $f_{ij} = g_i + h_j$ for each $i, j = 1, 2$?

(b) Andrey Nikolaevich and Vladimir Igorevich play the 'Dare you to decompose!' game. Some cells of chessboard are marked. A. N. writes numbers in the marked cells as he wishes. V. I. looks at the written numbers and chooses (as he wishes) 16 numbers $a_1, \dots, a_8, b_1, \dots, b_8$ as 'weights' of the columns and the lines. If each number in a marked cell turns out to be equal to the sum of weights of the line and the row (of the cell), then V. I. wins, and in the opposite case (i.e., when the number in at least one marked cell is not equal to the sum of weights of the line and the row) A. N. wins.

Prove that A. N. can win no matter how V. I. plays if and only if there does not exist a closed route of a rook turning only at marked cells (the route is not required to pass through each marked cell).

Let $\mathbb{R}^2$ be the plane with a fixed coordinate system. Let $x(a)$ and $y(a)$ be the coordinates of a point $a \in \mathbb{R}^2$. An ordered set (either finite or infinite) $\{a_1, \dots, a_n, \dots\} \subset \mathbb{R}^2$ is called an *array* if for each i we have $a_i \neq a_{i+1}$ and $x(a_i) = x(a_{i+1})$ for even i and $y(a_i) = y(a_{i+1})$ for odd i. It is not assumed that points of an array are distinct. An array is called *closed* if $a_1 = a_n$.

2. Consider a closed array $\{a_1, \dots, a_n = a_1\}$. A *decomposition* for such an array is an assignment of numbers at the projections of the points of the array on the x-axis and on the y-axis. Is it possible to put numbers $f_1, \dots, f_n \in \mathbb{R}$, where $f_1 = f_n$, at the points of the array so that for each decomposition there exists an f_i that is not equal to the sum of the two numbers at $x(a_i)$ and $y(a_i)$?

A subset $K \subset \mathbb{R}^2$ is called *discontinuously basic* if for each function $f : K \to \mathbb{R}$ there exist functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

3. (a) The segment $K = 0 \times [0, 1] \subset \mathbb{R}^2$ is discontinuously basic.

(b) The cross $K = 0 \times [-1, 1] \cup [-1, 1] \times 0 \subset \mathbb{R}^2$ is discontinuously basic.

4. (a) *A criterion for a subset of the plane to be discontinuously basic.* A subset of the plane is discontinuously basic if and only if it does not contain any closed arrays.

(b) Given a set of marked cells in the cube $8 \times 8 \times 8$, how can we see who wins in the 3D analogue of the 'Dare you to decompose!' game? In this analogue V. I. tries to choose 24 numbers $a_1, \dots, a_8, b_1, \dots, b_8, c_1, \dots, c_8$ so that the number at the cell (i, j, k) would be equal to the sum $a_i + b_j + c_k$ of the three weights.

(c)** Define discontinuous basic subsets of the 3-space. Discover and prove the 3D analogue of the above criterion.

Continuously basic subsets.

Denote by $|z,z_0| = |(x,y),\ (x_0,y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$ the ordinary distance between points $z=(x,y)$ и $z_0=(x_0,y_0)$ of the plane. Let K be a subset of $\mathbb{R}^2$. A function $f: K\to\mathbb{R}$ is called *continuous* if for each point $z_0\in K$ and number $\varepsilon>0$ there exists a number $\delta>0$ such that for each point $z\in K$ if $|z,z_0|<\delta$, then $|f(z)-f(z_0)|<\varepsilon$. It is sometimes convenient to write (x,y) instead of z.

5. (a) The function $f(x,y)=\sqrt{x^2+y^2}$ is continuous on the plane.

(b) The function $f(x,y)$ equal to the integer part of $x+y$, is not continuous on the plane.

(c) Let $a_1,\dots,a_n$ be distinct points of $K\subset\mathbb{R}^2$. Prove that there exists a continuous function $f: K\to\mathbb{R}$ such that $f(a_i)=(-1)^i$ and $|f(x)|\le 1$ for each $x\in K$.

(d) Let $K=\{a_1,\dots,a_{4n+4}\}$ be an array of $4n+4$ distinct points and $f_1,\dots,f_{4n+4}$ numbers such that $|(-1)^i-f_i|\le\frac{1}{2n}$. Let $g(x(a_i)), h(y(a_i)),\ i=1,\dots,4n+4$, be numbers such that $f_i=g(x(a_i))+h(y(a_i))$ for each i. Prove that $\max_i\{g(x(a_i))\}>n$.

In the sequel all functions are assumed to be continuous.

A subset $K\subset\mathbb{R}^2$ is called *(continuously) basic* if for each continuous function $f: K\to\mathbb{R}$ there exist continuous functions $g,h:\mathbb{R}\to\mathbb{R}$ such that $f(x,y)=g(x)+h(y)$ for each point $(x,y)\in K$.

6. (a) A closed array is not basic.

(b) The segment $K=0\times[0,1]\subset\mathbb{R}^2$ is basic.

(c) The cross $K=0\times[-1,1]\cup[-1,1]\times 0\subset\mathbb{R}^2$ is basic.

7. (a) If a subset of the plane is basic, then it is discontinuously basic.

(b) A *completed array* is the union of a point $a_0\in\mathbb{R}^2$ with an infinite array $\{a_1,\dots,a_n,\dots\}\subset\mathbb{R}^2$ of distinct points which *converges* to the point a_0 (i.e. for each $\varepsilon>0$ there exists a positive integer N such that for each $i>N$ we have $|a_i,a_0|<\varepsilon$). Prove that any completed array is not basic. (Note that it is discontinuously basic).

(c) Let $[a,b]$ be the rectilinear arc which connects points a and b. Prove that the cross $K=[(-1,-2),(1,2)]\cup[(-1,1),(1,-1)]$ is not basic.

(d) Let $m_{ij}=2-3\cdot 2^{-i}+j\cdot 2^{-2i}$. Consider the set of points $(x_{i,2l},x_{i,2l})$ and $(x_{i,2l-1},x_{i,2l-2})$, where i varies from 1 to ∞ and $l=1,2,3,\dots,2^{i-1}$. Prove that this subset of the plane does not contain any infinite arrays but contains arbitrary long arrays.

(e) The union of the set from the previous problem and the point $(2,2)$ is not basic.

8. Let $K\subset\mathbb{R}^2$ be the image of an arc $[0,1]$ under a continuous map $[0,1]\to\mathbb{R}^2$.

(a) Each continuous function $f: K\to\mathbb{R}$ assumes its lowest value and greatest value. Hint: reduce this problem to an analogous theorem on continuous functions $[0,1]\to\mathbb{R}$.

(b)* If K contains arbitrary long arrays, then K is not basic.

Hint. Assume that K contains arbitrary long arrays and is basic. We may assume that points of each array are distinct. Therefore for each n there is an array $\{a_1^n,\dots,a_{4n+4}^n\}$ of $4n+4$ distinct points in K. Then there exists continuous function $f_n: K\to\mathbb{R}$ such that $f_n(a_i^n)=(-1)^i$ and $|f_n(x)|\le 1$ for each $x\in K$. For each function $G: K\to\mathbb{R}$ its *maximum* is $\|G\|:=\max_{x\in K}|G(x)|$. Let $f: K\to\mathbb{R}$ and $g,h:\mathbb{R}\to\mathbb{R}$ be functions such that $\|f-f_n\|<1/2n$ and $f(x,y)=g(x)+h(y)$ for each $(x,y)\in K$. Then $\|g\|>n$...