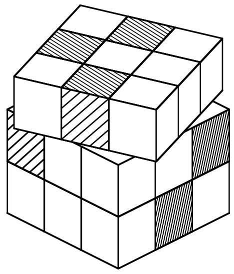
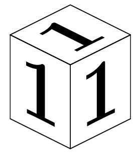




Rubic's cube and Higman problem

In this project we shall explore Rubic's cube and similar puzzles. Before this we should solve some preliminary problems.

Preliminary problems.

♦ **P1.** There are 12 labeled cubes in a row. The left cube is labeled with 1 and the right one with 12. If the crazy drummer beats his magic drumhead then two neighboring cubes transpose. After 333 beats every cube goes crazy if there are odd number of cubes with less number to the right of him. Can the number of the crazy cubes be equal to 6?

♦ **P2.** There are 42 cubes in the row labeled in the increased order. One can transpose every two neighboring cubes. Is it possible to transpose the first two cubes and leave the remaining cubes at their places by 2008 moves?

♦ **P3.** There are 20 color cubes in the circle. One can choose every three cubes and put the first one on the second one's place, second one to the third one's place and the third one to the first one's place. Is it possible to obtain the situation that all the cubes would be cyclically shifted in comparison with the initial position?

♦ **P4.** The facets of a cube are painted into different colors. Some of these cubes are used to form a rectangle $m \times n$. One can choose a row or a column and rotate all the cubes with regard to the common axis. Prove that it is possible to obtain the situation that all the cubes be rotated to the up with the same color.

A. Rubic's cube.

Let us call the whole large cube by *cube* and small cubes by *bricks*. Every 9-bricks facet of the cube can be rotated clockwise or counter-clockwise. One can do several such rotates in series. This sequence of rotates is simply called *combination*.

Denote the facets of the cube by A, B, C. We shall say that clockwise rotation of the facet is denoted by the same letter, for example, A. We also denote the counter-clockwise rotation by A^{-1}. Further, we shall denote the sequence of rotations by the sequence of the letters. For example, $ABA^{-1}C$ means *"clockwise rotation of the A facet then clockwise rotation of the B facet then counter-clockwise rotation of the A facet then clockwise rotation of the C facet".*

The sequence $XYX^{-1}Y^{-1}$ is called the *commutator* of the rotations X and Y.

There are three types of bricks inside the cube: *central* bricks are located in the centers of facets, corner bricks form the corners of the cube and middle ones are located in the centers of the cube's edges. In is clear that the central bricks doesn't move (with regard to each other). Also corner bricks will stay corner and middle ones will stay middle.

Suppose that middle and corner bricks don't fasten with anything. Hence we can easily take they out, transpose and put they in. Let us transpose middle bricks with other middle ones. Similarly we do with the corner ones. (We shall not transpose the central bricks.) Moreover, we check that the external facets of the bricks stay external after any transformation. Any position after such transformations is called a *state*. A state of the cube is called *right* if every facet of this cube is one-color. We shall say that a brick has the *right position* if it has the same colors of the facets as for right state of the whole cube. States are called *connected* if there exists a sequence of rotations such that the first state converts to the second one. A state is said to be *admissible* if it is connected with the right state.

♦ **A1.** Suppose that we apply a sequence of rotations to the initial cube. Prove that it is possible to apply this combination several times again and return to the initial state.

♦ **A2.** Is there exists a sequence of rotations that would arrange the cube starting from any state (by applying it several times)?

♦ **A3.** Find a combination of rotations that would cause to cyclically shift the bricks 1, 2, 3 and leave the remaining middle-bricks at their places (fig. 1).

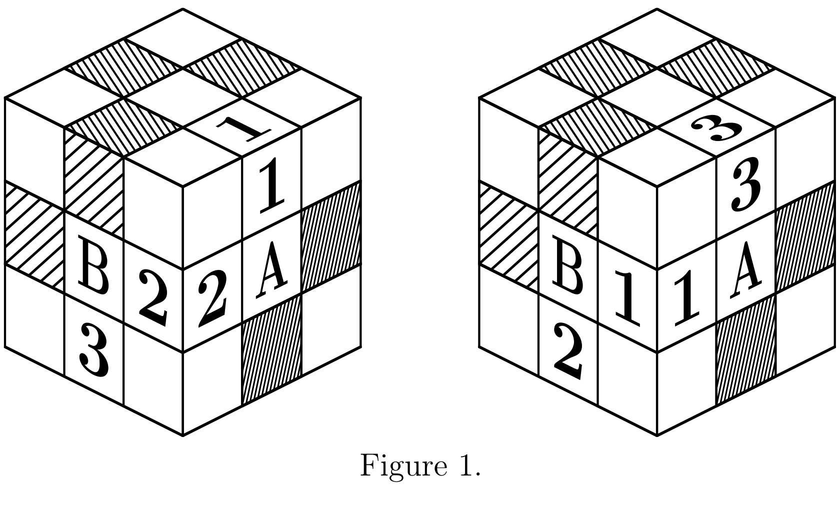


Figure 1.

♦ **A4.** Show that the combination $A^{-1}C^{-1}B^{-1}A^{-1}BAC$ cause to transpose the 1 and 2 bricks and leave the remaining middle-bricks at their places (fig. 2).

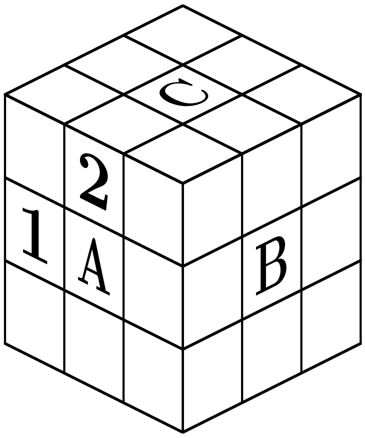


Figure 2.

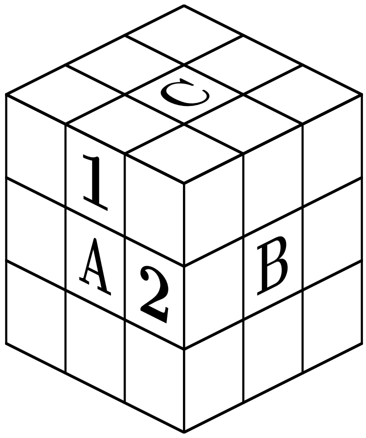


Figure 3.

♦ **A5.** Find a combination that would cause to rotate the bricks 1 and 2 within their sockets and leave the remaining middle-bricks at their places and states (fig. 3)?

♦ **A6.** Prove that there is no combination that would cause to rotate the brick 1 within its socket and leave the remaining middle-bricks at their places and states (fig. 3).

♦ **A7.** Suppose that the state of the cube is admissible. Show how to place all the middle bricks to the right states. Suppose that the state is not necessarily admissible. Consider the states of the middle bricks and describe all possible connected states.

♦ **A8.** Find a non-trivial combination of rotations that would cause no effect to the cube while applying it exactly three times.

♦ **A9.** Find a combination of rotations that would cause to cyclically shift the bricks 1, 2, 3 (fig. 4), and leave the remaining corner-bricks at their places and middle-bricks at their places and states.

♦ **A10.** Consider an admissible state of the cube. Find a combination that would arrange all the corner bricks to their places and leave the middle-bricks at their places and states. Suppose that the state is not necessarily admissible. Consider the states of the corner bricks and describe all possible connected states.

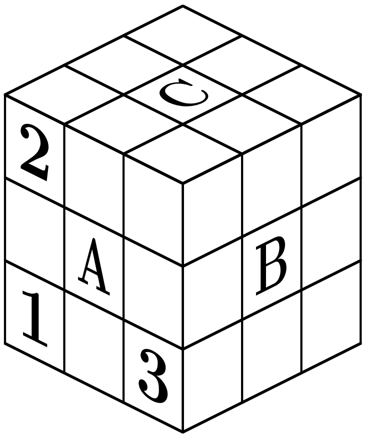


Figure 4.

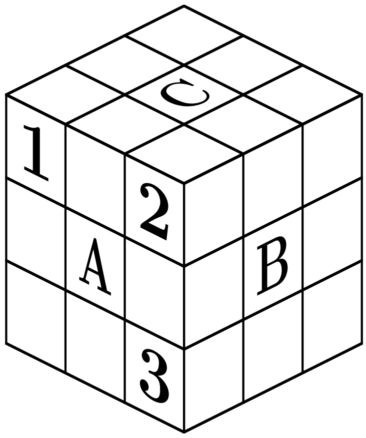


Figure 5.

♦ **A11.** Prove that there is no combination that would rotate exactly one corner brick and leave the remaining bricks at their places and states.

♦ **A12.** Find a combination that would cause to rotate clockwise the bricks 1, 2, 3 (fig. 5) by 120 degrees , and leave the remaining bricks at their places and states.

♦ **A13.** Suppose that the state of the cube is admissible. How to arrange the cube?

♦ **A14.** Suppose that the all corner bricks are in the right places and middle ones are in the right states. Consider the states of the corner bricks. How to define if it possible to arrange the cube?

♦ **A15.** How many pairwise unconnected states of the cube would be?

♦ **A16.** Calculate the number of admissible states of the cube.

Section B.

In this section we shall consider some similar puzzles. Let us apply two combinations (of rotations) at the same state. We shall say that these two combinations are *different* if they make different results.

♦ **B1.** The chessboard is labeled with all integer numbers from 1 to 64. One can choose a square 2×2 and rotate clockwise the numbers inside it. Prove that it is possible to achieve any possible numbers arrangement.

♦ **B2.** Consider the $2{\times}2{\times}2$ cube. Describe all the admissible states of this cube. How many such states are there?

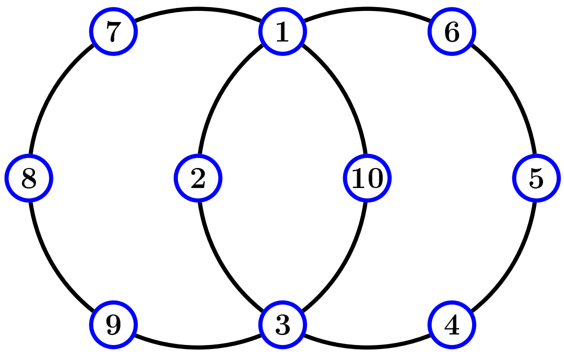


Figure 6.

♦ **B3.** Consider the game *"Hungarian rings"* (fig. 6). There is a planar puzzle consisting of two or more interwoven ovals each of which has several labeled pieces, some of which may belong to more than one oval. A puzzle move consists of shifting an oval by one or more *"increments"*, and hence all the pieces on it, along the oval's grooved track. The pieces are equally spaced apart (in spite of the typed depiction below) and whose pieces which lie on more than one oval can be moved along either oval. For simplicity, consider the puzzle consisting of only two ovals, each having 6 pieces.

The pieces 1 and 3 can be moved along either oval. Note that each move corresponds to an unique permutation of the numbers in $\{1, 2, \ldots, 10\}$.

Describe all the admissible states.

♦ **B4.** Consider the game *'Equator'* (fig. 7). This puzzle is in the shape of the sphere but has 3 circular bands encircling a sphere, each having 12 square-shaped pieces and each band intersecting each other at a 90 degree angle. Each pair of circles intersects at two points, or *"nodes"* and at each such node there is a puzzle piece shared by the two circular bands. There are 6 nodes total. One can rotate any band such that their pieces would transfer to each other. The total number of movable pieces is therefore $3 \times 12 - 6 = 30$.

Describe all the admissible states.

♦ **B5.** Consider the $4{\times}4{\times}4$ cube. Describe all the admissible states.

♦ **B6.** Consider the game "15". There are 15 numerated tiles placed in the $4{\times}4$ square. One square is empty. One can choose a neighboring (by side) tile of the empty square and move it on the empty place.

Describe all the admissible states of the game.

♦ **B7.** How many pairwise unconnected states of the cube $4{\times}4{\times}4$ would be?

♦ **B8.** Find the invariants system for $4{\times}4{\times}4$ cube.

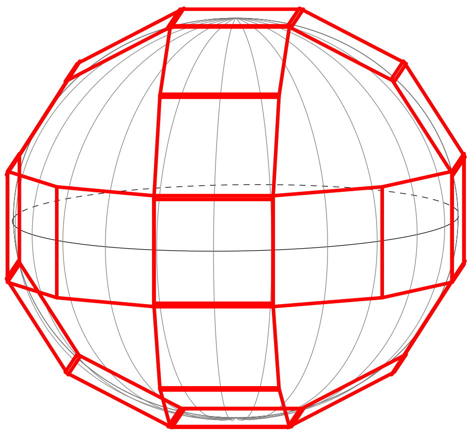


Figure 7.

♦ **B9.** Find the invariants system for $n{\times}n{\times}n$ cube.

Section C.

♦ **C1.** Consider a regular tetrahedra. We can rotate it such that it maps to itself and some edges and vertexes may transpose. Find the number of the different rotations.

♦ **C2.** Similar task for cube. Describe behavior of the main diagonals of cube.

Consider a regular polyhedra. Similarly to Rubic's cube rotations we can apply a sequence of two motions. It is easy to see that this composition is a motion itself. This composition is called a *product*.

The motion of polyhedra is called an *identity* if it preserves all its vertexes. Composition of every motion F with identical motion I does not change F, i.e. $F = IF = FI$. Identical motion of a polyhedra is an identity of all space.

♦ **C3.** Let M be an n-element set. Consider the permutations of M elements. For example, if M consists of 3 elements, then there are 6 such permutations:
1) exchange 1 and 2, 3 does not move;
2) exchange 2 and 3, 1 does not move;
3) exchange 1 и 3, 2 does not move;
4) sent 1 into 2, 2 into 3, 3 into 1 (cycle of length 3);
5) $1 \to 3$, $3 \to 2$, $2 \to 1$ (cycle of length 3);
6) don't move anything.

Similarly for each n all permutations can be listed. Suppose that we apply some permutation then apply another one. This combination of two permutations is called a *product* of them. Which permutation is identical? Prove that for any motion A there exists a motion A^{-1} such that $AA^{-1} = A^{-1}A = E$, there E is the identical motion. Check the associative law: $(AB)C = A(BC)$.

♦ **C4.** Let A be a set of all motions of a cube. Find a correspondence between A and the set B of permutations of 4 elements such that product in A corresponds to the product in B.

Definition. A *group* is a set G with operation of multiplication such that the following properties are satisfied:

1) $(AB)C = A(BC)$ (associative law) ;

2) there is an unit element E such that $AE = EA = A$ for any A;

3) for any element A there exists an inverse A^{-1} such that $AA^{-1} = A^{-1}A = E$.

Permutation group from problem C3 is denoted by S_n.

♦ **C5.** Prove that the following sets with operations are groups:
1) the set of integers with respect to addition;
2) the set of all positive rational numbers with respect to multiplication;
3) Rubic's cube transformation with respect to composition.

♦ **C6.** Which of the following objects are groups?
1) the rational numbers with respect to multiplication;
2) set of all words in the Latin alphabet (including empty word) with respect to concatenation (concatenation of u and v is uv).
3) set of all words in the alphabet $\{a, b, c\}$ (including empty word), if for any words X, Y we can replace any of the words $XabcY$, $XbcaY$, $XcabY$ with the word XY (in other words, we can remove abc, bca, cab from any word) and do inverse operation (add these words);
4) Three double transpositions (12)(34), (13)(24), (14)(23) and identity?
Note: (123)(4567) means that the elements are cyclically shifted in every bracket: $1 \to 2 \to 3 \to 1$ and $4 \to 5 \to 6 \to 7 \to 4$.

Definition. Let G be a group, H be a subset of G. Suppose that H contains unit element of G, (we just call it *unit*) and also all products of any two elements in H together with their inverses, then H is called a *subgroup*.

♦ **C7.** Let H be a subgroup of G. Prove that H is a group.

♦ **C8.** Find all subgroups of S_3.

♦ **C9. Lagrange theorem.** Prove that the number of subgroup elements divides the number of group elements.

♦ **C10.** Find a $\frac{n!}{2}$-elements subgroup of S_n, for $n \geq 2$.

We shall denote the group in the previous problem by A_n.

♦ **C11.** Prove that any element of A_n is a product of cycles of length 3 (they may intersect with each other).

Definition. $aba^{-1}b^{-1}$ is called a *commutator* of a and b.

Definition. *Commutant* of G is the set of all products of commutators.

♦ **C12.** Find the commutants of S_3, A_3, A_4, S_n, A_n.

Let us fix $a \in G$. For any $g \in G$ consider corresponding element $a^{-1}ga$. This element is called a *conjugate g respect to a*, or just *conjugate*.

♦ **C13.** Let H be a subgroup of G. Suppose that $a^{-1}Ha$ is the set of all conjugates of the H elements with respect to a. Prove that $a^{-1}Ha$ is a subgroup as well. Subgroups H and $a^{-1}Ha$ are called *conjugated*.

Definition. A subgroup is called *normal* if it is equal to all its conjugates.

♦ **C14.** Prove that commutant, unit element E and whole group G are normal subgroups of G.

The whole group G and unit element E are the *trivial* normal subgroups of G. All other subgroups are called *non-trivial*. If G has the trivial normal subgroups only, then G is *simple* group.

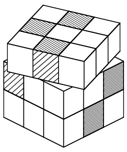
♦ **C15.** Find all n such that group S_n is simple.

♦ **C16.** Prove that group A_n is simple for $n \geq 5$.

♦ **C17.** Find all n such that S_n or A_n are equal to the groups of motions of cube or tetrahedra.

♦ **C18.** Prove that the group of icosahedra motions are equal to A_5.

♦ **C19.** Find a 8-element group such that there exist two elements a, b and $ab \neq ba$.



Section D. Groups

Definition. Groups G and H are called *isomorphic* if there exists a one-to-one mapping (*isomorphism*) φ such that unit element maps to unit element and composition of any two elements in G maps to composition of the corresponding two elements in H: $\varphi(g_1 g_2) = \varphi(g_1) \times \varphi(g_2)$.

♦ **D1.** Prove that the following groups are isomorphic:

1) group of cube's motions and S_4;

2) group of real numbers with respect to addition and group of horizontal motions;

3) group of integer numbers with respect to addition and group of 2^k for integer k with respect to composition.

4) group of dodecahedra's (icosahedra's) motions and A_5;

Isomorphic groups are really equivalent groups.

Suppose that G is a group, and M is a set. We shall say that G *acts* on M, if for any $m \in M$ and $g \in G$ there exists an corresponding element $m' = g(m)$ such that $(g_1 g_2)m = g_1(g_2 m)$.

Examples.

1. The group of all motions acts on 3-dimensional space.

2. The groups S_n и A_n act on the set $\{1, \ldots, n\}$.

3. A group acts on itself by *left multiplications*: every $h \in G$ defines a mapping $\varphi_h(g) = hg$.

4. A group acts on itself by conjugates: every $h \in G$ defines a mapping $\varphi_h(g) = h^{-1}gh$.

♦ **D2.** Prove that these are really the acts.

♦ **D3.** Suppose that n is the number of elements in the group G (we shall denote it by $n = |G|$). Prove that there exists a subgroup in S_n such that it is isomorphic to G.

♦ **D4.** A group G is colored into several colors such that the color of the product depends on colors of the factors only. The unity is red. Prove that the set of red elements is a normal subgroup.

Suppose that H is a subgroup in G. The coloring of G is called *left* if:

1) for any $g \in G$ and $h \in H$ the elements g and hg are colored in the same color;

2) if g_1 and g_2 are colored in the same color then $g_1 = hg_2$ for some $h \in H$.

♦ **D5.**

a) Prove that the left coloring coincides to the right one if and only if H is a normal group.

b) Prove that if H is a normal group then the color of the product depends on colors of the factors only and the color of the inverse element depends on color of the original one.

Suppose that H is a subgroup in G. For any $g \in G$ consider the set of all gh_i for $h_i \in H$. This set is called *left H-class with respect to H* and denoted by gH (note that this is essentially just the same as one-colored elements). Every element G belongs to one left H-class only. Two elements belong to the same class if they can be presented in the form gh_i for some $h_i \in H$. Every $g_1 = g_1e$ belongs to g_1H (because of $e \in H$). The element g_2 belongs to the same class if there exists h such that $g_2h = g_1$ (or $g_1g_2^{-1} = h$).

The product of two classes g_1H and g_2H is the class g_1g_2H. We can choose other representatives of these two classes, namely, $g_1' \in g_1H$ and $g_2' \in g_2H$. Then the product would be $g_1'g_2'H$.

♦ **D6.** Prove that a product is well-defined (because of $g_1g_2H = g_1'g_2'H$).

Similarly we can define an inverse element on the left H-classes set: the class $g^{-1}H$ is inverse for gH.

The set of left H-classes (or the set of colors from D5) can be considered as a group. This group is called a *factorgroup* with respect to the normal subgroup H.The left H-classes are the elements of this group.

♦ **D7.** Find factorgroups of G with respect to the H:

1) $G = S_n$ and $H = A_n$;

2) $G = A_4$ and H is a group of three double transpositions (12)(34), (13)(24), (14)(23) and identity?

3) G is a group of real numbers with respect to addition and H is the subgroup of integers.

♦ **D8.**

a) Prove that the group of rotations combinations of the cube 2×2×2 is the factorgroup of the 3×3×3-cube group.

b) Prove that the groups of rotations combinations of the cubes $3\times3\times3$ and $4\times4\times4$ is the factorgroups of the $5\times5\times5$-cube group.

Definition. An *orbit* of an element $m \in M$ is the set $\{g_i m\}$ for some $g_i \in G$.

⧫ **D9.** Describe the orbits of elements for the following acts:
1) the group of rotations combinations acts on the corner brick;
2) the horizontal translation group acts on the point of the plane;
3) group S_n of permutations acts by conjugates on itself. The element is 3-cycle.

⧫ **D10.** Prove that any two orbits are disjoint or coincide.

Definition. Стабилизатором элемента $m \in M$ называется множество элементов $g \in G$ таких, что $g(m) = m$.

Suppose that G acts on set M. Some elements of the group don't shift the elements of the set:

Definition. A set of elements $g \in G$ is called a *stabilizer* of en element $m \in M$ if $g(m) = m$.

⧫ **D11.** Prove that stabilizer $\operatorname{Stab} m$ is a subgroup. Prove that $|O_m| \cdot |\operatorname{Stab} m| = |G|$.

Definition. Действие называется *хорошим*, если стабилизатор любого элемента состоит из одной единицы.

Definition. An act is called *good* if stabilizer of any element is unity only.

Consider two actions of G on the sets M_1 и M_2. Suppose that these sets one-to-one correspond to each other: every element M_1 corresponds to the unique element M_2 and vice-versa: $\Psi(M_1) = M_2$. Then these two actions are called *conjugated* if the group acts on these sets by the similar way: element $g_\varphi(M_1)$ corresponds to $g_\psi(M_2)$.

Note that we can choose $M_1 = M_2 = M$.

⧫ **D12.**

a) Suppose that φ_1 and φ_2 are two good actions on M. Is it always true that they are conjugated?

b) The same question if the numbers of orbits are equal.

Suppose that G is a group and $g \in G$. Consider an element $g^{-1}hg$ for each g. It is easy to see that the element $g^{-1}h_1gg^{-1}h_2g = g^{-1}h_1h_2g$ correspond to the product h_1h_2. Hence, $h \to g^{-1}hg$ is one-to-one mapping such that a product maps to a product. Thus, we have obtain an isomorphism of G to itself.

♦ **D13.** Suppose that G is a group, H_1 and H_2 are isomorphic subgroups with isomorphism $\varphi : H_1 \to H_2$. Is it always possible to continue φ to the isomorphism of G to itself.

♦ **D14.** Suppose that G is a group and H_1 and H_2 are isomorphic subgroups of G. Let the numbers of elements in the H_1-classes and H_2 be equal (or both are countable). Prove that there exists a group $G^{'}$ such that G is a subgroup of $G^{'}$, $t \in G'$ and the following condition holds for any $h \in H_1$:

$$tht^{-1} = \varphi(h) \in H_2.$$

♦ **D15.** Suppose that the numbers of elements in the H_1-classes and H_2 not be equal. Prove the same fact.

Definition. If every element of group G can be presented by the product of some elements of $\{x_i\}$ then G is called *generated by elements* x_i. We denote this by $G = \langle x_i \rangle$.

Definition. A group is called *n-generated* if there are n elements in $\{x_i\}$.

Definition. A group is called *n-free-generated*, if it is isomorphic to the group of words in the alphabet $\{g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_n, g_n^{-1}\}$.

♦ **D16.** Suppose that G contains the following pairs of isomorphic subgroups: $\varphi_i : H_i \to H_i^{'}$ for $i = 1 \ldots n$. Prove that there exists a group $G^{'}$, such that G is a subgroup of $G^{'}$ and the following condition hold in G':

1) $t_i h_i t_i^{-1} = \varphi_i(h_i)$;
2) the group generated by $\langle t_i \rangle$ is free.

♦ **D17.** Suppose that group G is free-generated by the elements t_i $i \in \mathbb{N}$. Prove that there is a isomorphism between G and $H = \langle t_2, t_3, \ldots \rangle$.

♦ **D18.** Prove that any countable group can be presented by the subgroup of a 3-generated group.

Section E. Groups and arrangement of a high-dimensional cubes

We shall say that a cube is *almost solved* if all the bricks returned to their places but maybe some of them aren't rotated well. We use some following problems to almost solve a cube.

♦ **E1.** a) Suppose that two 4-cycle contain one common vertex. Prove that they generate a group S_7.

b) Suppose that two 4-cycle contain two common consecutive vertexes. Prove that they generate a group S_6.

♦ **E2.** a) Prove that the group A_{12} is generated by 11-cycles.

b) Let $x \in S_8$. Prove that $x^{8!}$ is the unity. Suppose that s is a 11-cycle. Prove that there exists a 11-cycle t such that $s = t^{8!}$.

c) Suppose that middle bricks and corner brick in the 3×3×3 cube are transposed with permutations of equal parity. Prove that it is possible to almost solve the cube.

d) Prove that it is possible to almost solve the cube 4×4×4 starting from any state.

Definition. We shall say that a group G is a *sum* of groups G_1 and G_2 if it is consists of pairs (g_1, g_2) such that $g_1 \in G_1$, $g_2 \in G_2$. Besides, the following rule describes the multiplication in G: $(g_1, g_2) \times (h_1, h_2) = (g_1h_1, g_2h_2)$. We denote a sum by $G = G_1 \oplus G_2$.

♦ **E3.** Suppose that G_1 is simple finite group with generators $a_1, \ldots, a_k$ and G_2 is simple finite group with generators $b_1, \ldots, b_k$. Let $G = G_1 \oplus G_2$. Suppose that H is a G-subgroup generated by the elements $z_i = (a_i, b_i), i = 1, \ldots, k$. Prove that $H = G$ or there is an isomorphism $\varphi : G_1 \to G_2$ such that $b_i = \varphi(a_i)$. Prove that the group of rotations combinations of the 3×3×3 cube contains a subgroup $A_8 \oplus A_{12}$.

♦ **E4.** Prove that any state in the cube $2n \times 2n \times 2n$ can be almost solved.

♦ **E5.** Prove that $2 \times 2 \times \ldots \times 2$ cube can be almost solved for any dimension.

♦ **E6.** Prove that $2n \times 2n \times \ldots \times 2n$ cube can be almost solved for any dimension.

Using the fact that a 2×2×2×2 cube can be almost solved we can do the useful note. *Suppose that we can rotate k corner bricks. Then we can similarly rotate any other k corner bricks.* This note helps us to describe a full cube solvability.

♦ **E7.** Prove that 2×2×2×2 cube has 3 classes of connected states in 4-dimensional space. Note: Use the fact that factorgroup of A_4 with respect to the group of double transpositions (example 2 in D7) is a 3-element group.

♦ **E8.** Prove that all states of the $2 \times 2 \times \ldots \times 2$ cube are admissible in the dimension 5 or higher.

♦ **E9.** Find the number of classes of connected states for $3 \times 3 \times \ldots \times 3$ cube in the dimension 4 or higher.

♦ **E10.** Find the number of classes of connected states for $n \times n \times \ldots \times n$ cube in the dimension 4 or higher.