On isogonal conjugacy, Miquel points, (Newton-)Gauss lines, etc.
Solutions.

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1 Introductory problems

1.
a)Follows from sine Ceva theorem.
b)Proof by counting angles.
c)Let P be a point of a given circle. Then measure of angle APB is fixed. Hence the sum of measures of angles CAP and CBP is fixed. Therefore, measure of angle $AP'B$ is fixed, hence P' lies on a fixed circle passing through A and B.
d)By P_A, P_B, P_C denote points symmetric to P in BC, CA, AB, respectively. Since $P_AC = PC = P_BC$, the perpendicular bisector of the segment P_AP_B passes through C, and hence it is the bisector CP' of angle P_ACP_B. Hence P' is the circumcenter of triangle $P_AP_BP_C$. By homothety with center P and ratio 1/2, the midpoint T of PP' is the center of the circle passing through projections of P to the sidelines. Similarly, T is the center of the circle passing through projections of P'. These two circles coincide since T is equidistant from projections of P and P' to a centain line.
In the case when P' is a point at infinity we obtain the Theorem on Simson line. for a point lying on the circumcircle, its projections of to the sidelines are collinear.
e)Let the common point of XY' and YX' be P and the common point of XY and $X'Y'$ be Q. Let $XA \cap YX' = X_A$, and $Y'A \cap YX' = Y_A$. Also call as Q_A intersection point of QA and YX'. Prove that lines AQ and AP are symmetric with respect to the bisector of angle A. Since this is true for all angles points P and Q are isogonaly conjugated.
Consider cross-ratio (Y, Q_A, X_A, X'). Project these points from Q to line AX'. The map of Y is X', the map of Q_A is A, X_A transforms to itself and X' transforms to the common point of lines $X'Y'$ and XA. Now project obtained ratio from Y' to line YX'. The map of A is Y_A, the map of X is P, X_A transforms to itself and the map of X' transforms back to X'. Thus $(Y, Q_A, X_A, X') = (P, Y_A, X_A, X') = (Y_A, P, X', X_A)$. It means that AQ is the reflection of AP in the bisector of angle A.
Another proof see also in the book of A.Akopyan and A.Zaslavsky "Geometry of conics" AMS, 2007.
f)By P' denote the point of intersection. Let AB and CD meet at K. For one of triangles KBC and KDA, P' is isogonal conjugate to P. Hence, P lies on the line symmetric to KP in the bisector of angle K. For the other of two triangles KBC and KDA, P' is also isogonal conjugate to P. We obtain that P lies on all four lines symmetric to AP, BP, CP, and DP, in the bisectors of corresponding angles.
g)Suppose that projections of P are concyclic. Then, similarly to p.d), obtain that point P' symmetric to P in the center of the circle is isogonal conjugate to P. The converse statement is proved anagolously.
h)See the book mentioned above.
i)By the statement converse to the Theorem on Simson line, it is sufficient to prove that the projections of parabola focus to the sidelines are collinear. Since a point symmetric to the focus in any tangent lies on a directrix, the required statement follows.

2.
a)Let the circumcircles of ABF and CDF meet for the second time at M. We have $\angle(AM, MD) = \angle(AM, MF) + \angle(MF, MD) = \angle(BA, BF) + \angle(CF, CD) = \angle(BA, CD) =$

$\angle(AE, ED)$, hence M lies on the circumcircle of ADE. Similarly for the other triangle.
b)It is sufficient to prove that triangles MBE and MFD are similar. We have $\angle(EB, BM) = \angle(CE, CM) = \angle(CD, CM) = \angle(FD, FM)$. Angles MEB and MDF are equal by the same reasoning.
c)Let B' and C' are the positions of bugs at some moment. The intersection point P of the circumcircles of triangles ABC and $AB'C'$ is the center of spiral similarity taking line BB' to CC' (ratio of this similarity is equal to the ratio of the speeds of the bugs). Hence P is Miquel point for quadruple of lines $B'B$, BC, CC', $C'B'$.
d)Suppose E moves from B to C at some constant speed, while F moves from D to A at the same speed. The condition holds at any moment. We show that each of points Q and R moves at a constant speed, so statement of the problem follows from the previous problem. Note that $\angle EQB = \angle FQD$, $EB = FD$. Hence the circumcircles of triangles EBQ FDQ are equal, and ratio EQ/QF is constant. Therefore Q moves at a constant speed. Similarly, for R.
e)Miquel point is the focus of a parabola inscribed ro the quadrilateral.
f)By Therem on Simson line, eah three of four projections are collinear. Hence all four projections are collinear. It is known that homothety with center at some point of the circumcircle and ratio 2 takes Simson line to the line passing through the orthocenter. Therefore, by homothety with center at Miquel point and ratio 2 we obtain the line l passing through orthocenters of four triangles. The orthocenters have equal powers with respect to three circles constructed on the diagonals of quadrilateral, hence l is the radical axis for these three circles that is perpendicular to Gauss line (passing through the centers of the circles). For the inscribed parabola, the focus lies on the circumcircles of the triangles. Hence, Miquel point is the focus. The projections of Miquel point lie on a tangent to the parabola at its vertex. The line passing through the orthocenters is the directrix of the parabola. Hence, Gauss line is parallel to the axis od parabola.

3.
a)By M denote the midpoint of AC, N — midpoint of BD, T — midpoint of EF. Let F', A' and B' are the midpoints of the sides of ABF. Note that M lies on $F'B'$, N lies on $F'A'$, T lies on $A'B'$. By homotheties with centers at vertices of ABF and ratio 2, $\frac{\overrightarrow{F'M}}{\overrightarrow{MB'}} = \frac{\overrightarrow{BC}}{\overrightarrow{CF}}$, $\frac{\overrightarrow{B'T}}{\overrightarrow{TA'}} = \frac{\overrightarrow{AE}}{\overrightarrow{EC}}$, $\frac{\overrightarrow{A'N}}{\overrightarrow{NF'}} = \frac{\overrightarrow{FD}}{\overrightarrow{DA}}$. Multiplying these three equalities, and applying Menelaus theorem, we obtain the required.
b)Hint. Gauss line of quadrilateral $ABCD$ is the locus of points X such that $S_{XAB} + S_{XCD} = S_{XBC} + S_{XDA}$ (here the areas are oriented).
c)Follows from 2f.
d)This is reformulation of b).

2 Three Miquels for a Quartet.

4. From Problems 13, 15 it follows that AX, BY, CZ pass through D' (isogonal conjugate to D with respect to triangle XYZ).

5. Follows directly from Problem 13.

6. Follows directly from Problems 13, 15.

7. By problem 11, $\triangle XDZ \sim \triangle XYA$, $\triangle XD'Z \sim \triangle XYA'$. Hence $XA : XD' = (XA : XZ)(XZ : XD') = (XY : XD)(XA' : XY) = XA' : XD$, that equivalent to the statement of the problem.

8. Follows from the previous problem and Theorem on three homotheties applied to the segments AA', DD', and $B'B$. Indeed, Z is the center of homothety that takes A to B', A' to B, etc. Alternative solution could be easily derived from Problem 1e).

9.
a)X is a center of a spiral similarity that takes C to D, A to B. Since K_Y is the image of L_Y under this spiral similarity, angle K_YXL_Y equals to $\angle(AC, BD)$. Hence X lies on the circle $P_YK_YL_Y$. Similarly obtain that Z lies on the same circle.
b)Since X lies on the circle AP_YB, we have $\angle XP_YB = \angle XAB$. Similarly, $\angle BP_YZ = \angle BCZ$. From these and four analogous equalities it follows that $\angle XP_YZ + \angle ZP_XY + \angle YP_ZX = \pi$, that implies the statement of the problem.
c)From problem 15 it is clear that $\psi_X(P_Y) = P_Z$, etc. This means that P_X, P_Y, P_Z belong to a certain quartet, so $\psi_X(P_X) = \psi_Y(P_Y) = \psi_Z(P_Z)$. We need to show that $\psi_Z(P_Z)$ is isogonal conjugate to T, or equivalently, $\psi_Z(T)$ is isogonal conjugate to P_Z. Note that ψ_Z takes circles ZXP_Y and ZYP_X (passing through T) to lines YP_X and XP_Y, hence $\psi_Z(T) = YP_X \cap XP_Y$. From equalities $\angle P_XYX = \angle ZYP_Z$, $\angle P_YXY = \angle ZXP_Z$ obtain the required statement.

3 Quartets for three Miquels.

10. a)-b) Follows directly from the definition.
c) Each of the transformations ψ_X, ψ_Y, ψ_Z is *circular* (i.e. takes a circle either to a circle or to a line) and preserves the orientation. Hence any product of ψ_Z, ψ_Y, ψ_X is a circular transformation preserving the orientation. Moreover, from a)-b) follows X, Y, Z are invariant points for $\psi_Z\psi_Y\psi_X$. Note that a circular transformation preserving the orientation is uniquely defined by the images of three points. The statement of the problems is independent of the order of ψ_X, ψ_Y, ψ_Z in its product. Hence ψ_X, ψ_Y, ψ_Z commute to each other.
11. By definition of ψ_X, $\angle ZXD = \angle AXY$ and $XD \cdot XA = XY \cdot XZ$, that implies the first similarity. The second similarity is proved analogously.
12. From Problem 10 it follows, in particular, that $\psi_X \circ \psi_Y = \psi_Z^{-1} = \psi_Z$. Therefore, $\psi_Y(A) = \psi_Y(\psi_X(D)) = \psi_Z(D) = C$, so ψ_Y interchanges the points in the pairs (A, C), (B, D). Similarly we obtain that ψ_X interchanges the points in the pairs (A, D), (B, C), while ψ_Z interchanges the points in the pairs (A, B), (C, D).
13. Let D', A' be isogonal conjugates to D, A respectively. Then A' lies on XD, D' lies on XA. Moreover, $\angle XD'Z = \pi - \angle ZXD' - \angle D'ZX = \pi - \angle DXY - \angle YZD = \angle ZDX + \angle XYZ - \pi$. By Problem 11 we have $\angle ZDX = \angle AYX$, i.e. $\angle XD'Z = \angle XYA'$. Hence $\triangle XD'Z \sim \triangle XA'Y$, and $A' = \psi_X(D')$.
14.
a)The incenter and the excenters of XYZ.
b)Point O and three points symmetric to X, Y, Z in the opposite sidelines of XYZ.

c)From the definition of ψ_Z it follows that its invariant point lie on bisector of angle XZY, and on the circle with center Z and radius $\sqrt{ZX \cdot ZY}$. There are two such points U and V. From Problem 10 we obtain that $\psi_X(U) = \psi_Y(\psi_Z(U)) = \psi_Y(U) = \psi_Z(\psi_X(U))$, i.e. $\psi_X(U)$ is also an invariant point for ψ_Z. It is clear that $\psi_X(U) \neq U$, therefore, $\psi_X(U) = \psi_Y(U) = V$, and a required quartet (both for U and for V) is U, U, V, V. Further, the isogonal conjugates to U, V are also invariant under ψ_Z. Hence U and V are isogonal conjugates to each other, and quartet U, U, V, V coincides to conjugate quartet.
15.
a)From the definition of ψ_X and the Problem 12 it follows that X is the center of spiral similarity taking A to B, and C to D. By Problem 2b) this center is Miquel point.
b)Y is Miquel point for the quadruple of lines AB, BC, AD, CD; Z is Miquel point for the quadruple AD, AC, BD, BC.
c)X is Miquel point, hence (in particular) $\triangle XBD \sim \triangle XAC$. Let $P_X = AD \cap BC$, $P_Y = AC \cap BD$, $P_Z = AB \cap CD$. We have $XA \cdot XD = XB \cdot XC = XP_Y \cdot XP_Z = R_a^2$, and the angles AXD, BXC, P_YXP_Z have a common bisector l. The inversion with center X and radius R_a followed by the symmetry in l takes triangles ADP_Y and BCP_Y to DAP_Z and CBP_Z, respectively. Therefore it tales intersection point Y of the circles ADP_Y and BCP_Y to Z. Hence the product of an inversion and a symmetry defined above is ψ_X.
16. From Problem 13 it follows that the product of ψ_X and the isogonal conjugacy (in any order) takes D to A'.