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Colorings and clusters

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There is **a mistake** in the first version of the problem 16. Here is the right version.

16. Show that the statement of the problem 12 can be obtained from the following fact, which is base for topological definition of dimension: if n-dimensional space is covered by some **open** sets of bounded diameter then there exists a point covered $n+1$ times.

18. Suppose the layer between two parallel lines is colored by two colors. Prove that there exist 2 points of the same color inside the layer such that $distance(A, B) = 1$.

19.* The same problem for the layer between two parallel planes, colored by 4 colors.

20. a) **Sperner Lemma.** Triangle ABC is divided by small triangles; vertices of these triangles are colored by 3 colors such that A is colored by color 1, B by color 2, C by color 3. Vertices on $[AB]$ are colored by colors 1 or 2; vertices on $[BC]$ – by colors 2 or 3, vertices on $[CA]$ — by colors 3 or 1. Prove that there exists a small triangle, which vertices are colored by different colors.

b) Generalize this lemma for n-dimensional space (consider also $n = 1$).

21. a) Prove that there is no continuous mapping of disc onto its boundary R such that it is identical on the border. This mapping is called *Retract*.

b) Prove the **Brauer theorem:** Continuous mapping F of the disc into itself has a stable point, i.e. there exists a point x_0 such that $F(x_0) = x_0$.

c) Generalize this for n-dimensional space.

22. **Inductive definition of dimension.** 0-dimensional space is a space such that any 2 points are situated in different connected components, 1-dimensional space is a space such that any 2 points can be separated by 0-dimensional space (and it is not 0-dimensional). n-dimensional space is a space, such that any two points are separated by $n-1$-dimensional space (and it is not $(n-1)$-dimensional). Prove that $\mathbb{R}^n$ is n-dimensional space according this definition.