

# Tiling by rectangles and alternating current

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## Abstract

This paper is on tilings of polygons by rectangles. A celebrated physical interpretation of such tilings due to R.L. Brooks, C.A.B. Smith, A.H. Stone and W.T. Tutte uses direct-current circuits. The new approach of the paper is an application of alternating-current circuits. The following results are obtained:

- a necessary condition for a rectangle to be tilable by rectangles of given shapes;
- a criterion for a rectangle to be tilable by rectangles similar to it but not all homothetic to it;
- a criterion for a generic polygon to be tilable by squares.

These results generalize the ones of C. Freiling, R. Kenyon, M. Laczkovich, D. Rinne and G. Szekeres.

*Keywords:*

Tiling, rectangle, orthogonal polygon, alternating current

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## 1. Introduction

A rectangle  $a \times b$ , where  $a$  and  $b$  are integers, can be tiled by  $a \cdot b$  squares. Thus a rectangle with rational side ratio can be tiled by squares. In 1903 M. Dehn proved the converse assertion:

**Theorem 1.1.** [10] *A rectangle can be tiled by squares (not necessarily equal) if and only if the ratio of two orthogonal sides of the rectangle is rational.*

Although this assertion is expectable, the proof is complicated. After original proof, many improvements have been made [2, 3, 18, 25, 32].

The most interesting for us is the approach of R.L. Brooks, C.A.B. Smith, A.H. Stone and W.T. Tutte [3]. To a tiling of a rectangle they assign a direct-current circuit, and then deduce Theorem 1.1 from certain properties of the circuit. They also apply the technique to find a tiling of a square by squares of distinct sizes, see the figure in the front cover of the journal [13].

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We study finite tilings by arbitrary nondegenerate rectangles. The sides of rectangles are assumed to be parallel to coordinate axes, i.e., either vertical or horizontal. By *the ratio* of a rectangle we mean the length of the horizontal side divided by the length of the vertical one. We study the following problem posed in [16, p. 218] and [19, p. 3]:

**Problem 1.2.** Which rectangles can be tiled by rectangles of given ratios  $c_1, \dots, c_n$ ?

A related problem of *signed* tilings is solved in [19].

For  $n = 1$  and  $c_1 = 1$  the question of Problem 1.2 is answered by Theorem 1.1. A necessary condition for arbitrary  $n$  was actually proved by M. Dehn: if a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$  then  $c$  is (the value of) a rational function in  $c_1, \dots, c_n$  with rational coefficients.

This function depends only on "combinatorial structure" of the tiling. For instance, if a rectangle of ratio  $c$  is dissected into 2 rectangles of ratios  $c_1$  and  $c_2$  by a vertical (respectively, horizontal) cut then  $c(c_1, c_2) = c_1 + c_2$  (respectively,  $c(c_1, c_2) = \frac{c_1 c_2}{c_1 + c_2}$ ). The problem reduces to description of possible functions  $c(c_1, \dots, c_n)$ . By the mentioned physical interpretation this is equivalent to a natural problem: *describe possible formulas  $c(c_1, \dots, c_n)$  expressing the conductance of a planar direct-current circuit through the conductances  $c_1, \dots, c_n$  of individual resistors.*

The main idea of the paper is to apply *alternating-current* circuits (equivalently, circuits with complex-valued conductances) to the above problems. Our first result is

**Theorem 1.3.** *Suppose that a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$ . Then  $c = C(c_1, \dots, c_n)$  for some rational function  $C(z_1, \dots, z_n)$  such that*

- (1)  $C(z_1, \dots, z_n)$  has rational coefficients, i.e.,  $C(z_1, \dots, z_n) \in \mathbb{Q}(z_1, \dots, z_n)$ ;
- (2)  $C(z_1, \dots, z_n)$  is degree 1 homogeneous, i.e.,  $C(tz_1, \dots, tz_n) = tC(z_1, \dots, z_n)$ ;
- (3) if  $\operatorname{Re} z_1, \dots, \operatorname{Re} z_n > 0$  then  $\operatorname{Re} C(z_1, \dots, z_n) > 0$ .

**Problem 1.4.** Is the converse theorem true for  $n \geq 3$ ?

Parts (1) and (2) of Theorem 1.3 were actually proved by Dehn, see also [17, Lemma 4]. Case  $n = 1$  (respectively,  $n = 2$ ) of both Theorem 1.3 and its converse is equivalent to Theorem 1.1 (respectively, to [16, Theorem 5], see also Theorem 3.1 below). For  $n \geq 3$  the converse theorem cannot be proved by our method, see Example 3.2.

Theorem 1.3 has a clear physical meaning, see §2.4. But this theorem (even together with its converse) is not *algorithmic*, i.e., it does not give an algorithm to decide if there exists a required tiling. Thus it is interesting to get less general but algorithmic results.

A result of this kind was obtained independently by C. Freiling, D. Rinne in 1994 and M. Laczkovich, G. Szekeres in 1995. It uses the following notion. An *algebraic conjugate* of an algebraic number  $c$  is a complex root of the minimal integral polynomial of  $c$ .

**Theorem 1.5.** [17, 22] *For  $c > 0$  the following 3 conditions are equivalent:*

- (1) *a square can be tiled by rectangles of ratios  $c$  and  $1/c$ ;*
- (2) *the number  $c$  is algebraic and all its algebraic conjugates have positive real parts;*
- (3) *for certain positive rational numbers  $d_1, \dots, d_m$  we have*

$$d_1 c + \frac{1}{d_2 c + \dots + \frac{1}{d_m c}} = 1.$$

We present a new short self-contained proof of this result. This new proof is an example of a natural application of alternating-current circuits. We also get a new algorithmic result:

**Theorem 1.6.** *For a number  $c > 0$  the following 3 conditions are equivalent:*

- (1) *a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c$  and  $1/c$  (in such a way that there is at least one rectangle of ratio  $1/c$  in the tiling);*
- (2) *the number  $c^2$  is algebraic and all its algebraic conjugates distinct from  $c^2$  are negative real numbers.*
- (3) *for certain positive rational numbers  $d_1, \dots, d_m$  we have*

$$\frac{1}{d_1c + \frac{1}{d_2c + \dots + \frac{1}{d_m c}}} = c.$$

More algorithmic results can be found in [16, p. 224]. For similar results on tiling by triangles see [29]. For higher dimensional generalizations see [25].

We also consider tilings of arbitrary (not necessarily convex) polygons by rectangles. This generalization reveals new connections between tilings and electrical circuits.

We apply direct-current circuits with several terminals to get a criterion for a generic polygon to be tilable by squares (Theorem 4.2 below, again not algorithmic). This result generalizes Theorem 1.1 and [21, Theorems 9 and 12]. An easier related problem of *signed* tiling by squares is solved in [15, 20].

We apply alternating-current circuits with several terminals to get a short proof of a generalization of Theorem 1.5 to polygons with rational vertices [28] (Theorem 4.3 below). We also give basic results on *electrical impedance tomography* for alternating-current circuits, cf. [9, 6, 7, 23].

There is a close relationship among electrical circuits, discrete harmonic functions and random walks on graphs [11, 24, 1]. Our results have equivalent statements in the language of each of the theories, e.g., see Corollary 4.9 below.

The paper splits naturally into two formally independent parts: §§1–3 and §§4–6.

The first part contains the proof of Theorems 1.3, 1.5 and 1.6. In §2 the basics of electrical circuits and their connection with tilings are recalled. In §3 the results of §1 are proved.

The second part concerns some variations. In §4 the results on tilings of polygons, electrical impedance tomography and random walks are stated. In §5 the results of §2 are generalized to electrical circuits with several terminals. In §6 the results of §4 are proved.

## 2. Main ideas

### 2.1. Electrical circuits

Our approach is based on electrical circuits theory [26]. However, the reader is not assumed to be familiar with physics. In this section we recall all the required physical concepts (although the presentation is formal and physical meaning is explained very briefly). This section does not contain new results. For short proofs see §5.

An *electrical network* is a connected graph with a nonnegative real number (*conductance*) assigned to each edge, and two marked (*boundary*) vertices.

For simplicity assume that the graph does not have neither multiple edges nor loops. Although all the concepts below can be adopted easily for the graphs with multiple edges. We say that electrical network is *planar* if the graph is drawn in the unit disc in such a way that the boundary vertices are in the boundary of the disc and the edges do not intersect each other.

Fix an enumeration of the vertices  $1, 2, \dots, n$  of the graph such that 1 and 2 are the boundary ones. It is convenient to denote the number of boundary vertices by  $b = 2$ . Let  $m$  the number of edges. Denote by  $c_{kl}$  the conductance of the edge between the vertices  $k$  and  $l$ . Set  $c_{kl} = 0$  if there is no edge between  $k$  and  $l$  in the graph.

An *electrical circuit* is an electrical network along with two real numbers  $U_1$  and  $U_2$  (*incoming voltages*) assigned to the boundary vertices.

Each electrical circuit gives rise to certain numbers  $U_k$ , where  $1 \leq k \leq n$  (*voltages* at the vertices), and  $I_{kl}$ , where  $1 \leq k, l \leq n$  (*currents* through the edges). These numbers are defined by the following axioms:

(C) *The Ohm law.* For each pair of vertices  $k, l$  we have  $I_{kl} = c_{kl}(U_k - U_l)$ .

(I) *The Kirchhoff current law.* For each vertex  $k > b$  we have  $\sum_{l=1}^n I_{kl} = 0$ .

Informal meaning of law (I) is that electrical charge is not aggregated at the nonboundary vertices. In other words, these laws assert that  $U_k$  is a *discrete harmonic function*. The numbers  $U_k$  and  $I_{kl}$  are well-defined by these axioms by the following classical result.

**Theorem 2.1.** [31] *For any electrical circuit the system of linear equations (C),(I) in variables  $U_k$ ,  $b < k \leq n$ , and  $I_{kl}$ ,  $1 \leq k, l \leq n$ , has a unique solution.*

Denote by  $I_1 = \sum_{k=1}^n I_{1k}$  the current flowing inside the circuit through vertex 1. *The conductance* of an electrical circuit with  $U_1 \neq U_2$  is the number  $C = I_1/(U_1 - U_2)$ . Clearly, the conductance does not depend on  $U_1$  and  $U_2$ . Thus *the conductance* of an electrical network is well-defined. Basic examples of networks and their conductances are shown in figure 1.

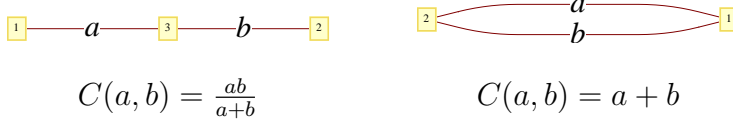


Figure 1: Series and parallel electrical networks

## 2.2. Tilings and networks

There is a close relationship between electrical networks and tilings. We say that an edge  $kl$  of a circuit is *essential*, if  $I_{kl} \neq 0$ . Clearly, the property of an edge being essential does not depend on  $U_1$  and  $U_2$  if  $U_1 \neq U_2$ .

**Lemma 2.2.** [3, 4, Theorem 1.4.1] *The following two conditions are equivalent:*

- (1) *a rectangle of ratio  $c$  can be tiled by  $m$  rectangles of ratios  $c_1, \dots, c_m$ ;*
- (2) *there is a planar electrical network having conductance  $c$  and consisting of  $m$  essential edges of conductances  $c_1, \dots, c_m$ .*

Let us sketch the proof of assertion (1)  $\implies$  (2). Given a tiling as in (1) construct an electrical network as follows (see figure 2). Take a point in each maximal horizontal cut of the tiling and in each horizontal side of the tiled rectangle. These points are vertices of the network. For each rectangle in the tiling draw an edge between the vertices in the cuts containing the horizontal sides of the rectangle. Set the conductance of the edge to be the ratio of the rectangle. The obtained network has conductance  $c$ , see §5.2 for the proof.

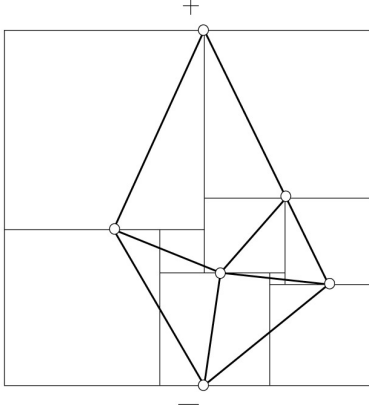


Figure 2: Correspondence between tilings and electrical networks

### 2.3. Formulas for conductance

Let us summarize some useful properties of formulas for conductance.

**Lemma 2.3.** *Suppose that an electrical network consists of  $m$  edges of conductances  $c_1, \dots, c_m$ . Then the conductance of the network  $C(c_1, \dots, c_m)$  has the following properties:*

- (1) [3]  $C(c_1, \dots, c_m) \in \mathbb{Q}(c_1, \dots, c_m)$ ;
- (2) [3]  $C(c_1, \dots, c_m)$  is degree 1 homogeneous;
- (3) [3]  $\frac{\partial}{\partial c_j} C(c_1, \dots, c_m) = \left( \frac{U_k - U_l}{U_1 - U_2} \right)^2$ , where  $k$  and  $l$  are the endpoints of the edge  $j$ ;
- (4) [27] if  $c_1, \dots, c_m > 0$  then  $\frac{\partial}{\partial c_j} C(c_1, \dots, c_m) \geq 0$ ; if the edge  $j$  is essential then the latter inequality is strict;
- (5) [5] if  $\text{Re } c_1, \dots, \text{Re } c_m > 0$  then  $\text{Re } C(c_1, \dots, c_m) > 0$ .

*Remark 2.4.* (A. Akopyan, private communication) Property (4) follows from (1), (2) and (5). Property (5) does not follow from (1), (2) and (4), e.g., the function  $C(c_1, c_2) = (c_1 + c_2) \frac{c_1^2 + c_2^2}{c_1^2 + 2c_2^2}$  satisfies (1), (2), (4) but not (5).

Property (5) concerns the extension of the function  $C(c_1, \dots, c_m)$  to the complex plane. This fundamental property does not seem to be paid attention for *direct-current* circuits. Certainly it is well-known for *alternating-current* circuits. Short proof of the lemma is given in §5.1.

### 2.4. Alternating-current circuits

Let us explain informal physical meaning of fundamental Lemma 2.3(5) and condition (3) of Theorem 1.3. This is not used elsewhere in the paper and the reader may easily skip this subsection.

Informally, an *alternating-current circuit* is a collection of conductors, condensers, inductors and a single alternating-voltage source connected with each other.

Formally, an *alternating-current circuit* is a graph with the following structure:

- two marked (*boundary*) vertices;
- two functions (*voltages*)  $\tilde{U}_1(t) = U \cos \omega t$  and  $\tilde{U}_2(t) = 0$  assigned to them;
- division the edges into three types (*conductors, condensers and inductors*);
- a positive number  $\tilde{c}_{kl}$  assigned to each edge (called *conductance, capacitance or inductance*, depending on the type of the edge).

The *voltages*  $\tilde{U}_k(t)$  and the *currents*  $\tilde{I}_{kl}(t)$  are defined by the following axioms:

( $\tilde{C}$ ) *The generalized Ohm law.* For each edge  $kl$  we have

$$\tilde{I}_{kl}(t) = \begin{cases} \tilde{c}_{kl}(\tilde{U}_k(t) - \tilde{U}_l(t)) & \text{if } kl \text{ is a conductor;} \\ \tilde{c}_{kl} \frac{d}{dt}(\tilde{U}_k(t) - \tilde{U}_l(t)) & \text{if } kl \text{ is a condenser;} \\ \tilde{c}_{kl} \int_{\pi/2\omega}^t (\tilde{U}_k(t) - \tilde{U}_l(t)) dt & \text{if } kl \text{ is an inductor.} \end{cases}$$

( $\tilde{I}$ ) *The Kirchhoff current law.* For each vertex  $k \neq 1, 2$  we have  $\sum_{l=1}^n \tilde{I}_{kl}(t) = 0$ .

The voltages and the currents can be found using the following well-known algorithm. Denote by  $i = \sqrt{-1}$ . Put  $U_1 = U$ ,  $U_2 = 0$  and

$$c_{kl} = \begin{cases} \tilde{c}_{kl}, & \text{if } kl \text{ is a conductor;} \\ i\omega \tilde{c}_{kl}, & \text{if } kl \text{ is a condenser;} \\ \frac{1}{i\omega} \tilde{c}_{kl}, & \text{if } kl \text{ is an inductor.} \end{cases}$$

Define the complex numbers  $U_k$ ,  $3 \leq k \leq n$ , and  $I_{kl}$ ,  $1 \leq k, l \leq n$ , by *direct-current* laws (C), (I). Then  $\tilde{U}_k(t) = \operatorname{Re}(U_k e^{i\omega t})$ ,  $\tilde{I}_{kl}(t) = \operatorname{Re}(I_{kl} e^{i\omega t})$ . In this sense alternating-current circuits are "equivalent" to direct-current circuits with complex-valued conductances (also called *admittances*).

Notice that always  $\operatorname{Re} c_{kl} \geq 0$ . Physically this means nonnegative *energy dissipation* at the edge  $kl$  (which is  $\operatorname{Re} c_{kl} |U_k - U_l|^2$ ). Thus a physical meaning of Lemma 2.3(5) is: "a network consisting of elements dissipating energy also dissipates energy".

### 2.5. Positive real functions

This subsection is used in the proof of only assertions (2)  $\implies$  (3) in Theorems 1.5 and 1.6.

Consider electrical circuits, in which all the edges have conductances  $z$  and  $1/z$ ,  $\operatorname{Re} z > 0$ . (They have a natural physical meaning: circuits consisting of condensers and inductors with incoming voltage of complex frequency  $z/i$ .) Let us describe possible conductances  $C(z)$  of such electrical circuits. By Lemma 2.3(1), (2) and (5) the functions  $C(z)$  are *positive real*, i.e., satisfy condition (1) of the following lemma. Denote by  $\operatorname{Re} \infty = 0$ ,  $C(\infty) = \lim_{z \rightarrow 0} C(1/z)$  and  $C'(\infty) = \lim_{z \rightarrow 0} (C(1/z))'$ .

**Lemma 2.5.** [5, 14, 16, Lemma 4] *For an odd function  $C(z) \in \mathbb{R}(z)$  the following 5 conditions are equivalent:*

- (1) *if  $\operatorname{Re} z > 0$  then  $\operatorname{Re} C(z) > 0$ ;*
- (2) *if  $C(z) = 1$  then  $\operatorname{Re} z > 0$ ;*
- (3) *if  $C(z) = 0$  then  $\operatorname{Re} z = 0$  and  $C'(z) > 0$  (here  $z \in \mathbb{C}$  or  $z = \infty$ );*
- (4) *either  $C(z)$  or  $1/C(z)$  equals*

$$d_1 z \prod_{k=1}^n \frac{z^2 + a_k^2}{z^2 + b_k^2},$$

*for some integer number  $n \geq 0$  and real numbers  $d_1 > 0$ ,  $a_1 > b_1 > a_2 > \dots > b_n \geq 0$ ;*

- (5) *either  $C(z)$  or  $1/C(z)$  equals*

$$d_1 z + \frac{1}{d_2 z + \dots + \frac{1}{d_m z}},$$

*for some integer number  $m \geq 1$  and real numbers  $d_1, \dots, d_m > 0$ .*

Parts of the lemma are proved in [5, 14] and in [16] using the results of [30]. A short proof is given in §5.3.



### 3. Proof of main results

#### 3.1. Proof of Theorem 1.3

Hereafter in an *electrical circuit* or *network* we allow the conductances to be arbitrary complex numbers with positive real part. This generalization of the above notion is motivated by §2.4 (and describes both direct- and alternating-current circuits). Theorem 1.3 is an easy consequence of the results of §2:

*Proof of Theorem 1.3.* Suppose that a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$ . By Lemma 2.2 there is an electrical network of conductance  $c$  consisting of edges of conductances  $c_1, \dots, c_n$ . For each  $k = 1, \dots, n$  replace each edge of conductance  $c_k$  in the network by an edge of complex conductance  $z_k$ ,  $\operatorname{Re} z_k > 0$ . Let  $C(z_1, \dots, z_n)$  be the conductance of the obtained network. The function  $C(z_1, \dots, z_n)$  has the properties (1)–(3) of Theorem 1.3 by Lemma 2.3(1),(2) and (5).  $\square$

#### 3.2. Proof of Theorem 1.5

*Proof of Theorem 1.5.* (3)  $\implies$  (1) [16] Suppose that condition (3) of Theorem 1.5 holds and, say,  $m$  is odd. Take a unit square. Cut off a rectangle of ratio  $d_1 c$  from the square by a vertical cut. The remaining part is a rectangle of ratio

$$1 - d_1 c = \frac{1}{d_2 c + \dots + \frac{1}{d_m c}}.$$

Now cut off a rectangle of ratio  $1/d_2 c$  from the remaining part by a horizontal cut. We get a rectangle of ratio

$$d_3 c + \frac{1}{d_4 c + \dots + \frac{1}{d_m c}}.$$

Continue this process alternating vertical and horizontal cuts. Condition (3) guaranties that after step  $(m - 1)$  we get a rectangle of ratio  $d_m c$ . We obtain a tiling of the square by rectangles of ratios  $d_1 c, 1/d_2 c, d_3 c, 1/d_4 c, \dots, d_m c$ . Since all  $d_k \in \mathbb{Q}$  one can chop the tiling into rectangles of ratios  $c$  and  $1/c$ .

(1)  $\implies$  (2). Suppose that a square is tiled by rectangles of ratios  $c$  and  $1/c$ . By Lemma 2.2 there exists an electrical network of conductance 1 with edge conductances  $c$  and  $1/c$ . Replace each edge of conductance  $c$  (respectively,  $1/c$ ) in this network by an edge of conductance  $z \in \mathbb{C}$  (respectively,  $1/z$ ). Let  $C(z)$  the conductance of the obtained network. Then  $C(z) \in \mathbb{Q}(z)$  by Lemma 2.3(1).

Since  $C(c) = 1$  it follows that  $c$  is algebraic ( $C(z)$  is nonconstant because  $C(-c) = -C(c) = -1$  by Lemma 2.3(2)). Let  $z$  be an algebraic conjugate of  $c$ . Then still  $C(z) = 1$ .

Let us prove that  $\operatorname{Re} z > 0$ . Indeed, first assume that  $\operatorname{Re} z < 0$ . Then  $\operatorname{Re}(-z) > 0$  and  $\operatorname{Re}(-1/z) > 0$ . Thus by Lemma 2.3(5) we have  $0 < \operatorname{Re} C(-z) = -\operatorname{Re} C(z) = -1$ , a contradiction. Now assume that  $\operatorname{Re} z = 0$ . Let  $z_k \rightarrow z$ , where each  $\operatorname{Re} z_k < 0$ . Still  $0 < \operatorname{Re} C(-z_k) = -\operatorname{Re} C(z_k) \rightarrow -1$ , a contradiction. Thus  $\operatorname{Re} z > 0$ .

(2)  $\implies$  (3) [16] Let  $p(z)$  be a minimal polynomial of  $c$ . Put  $C(z) = \frac{p(-z) - p(z)}{p(-z) + p(z)}$ . Then  $C(c) = 1$ ,  $C(z) \in \mathbb{Q}(z)$ ,  $C(z)$  is odd and all the roots of the equation  $C(z) = 1$  have positive real part. By Lemma 2.5(2)  $\implies$  (5) the function  $C(z)$  satisfies condition (5) of Lemma 2.5. Since  $C(z) \in \mathbb{Q}(z)$  it follows by Euclidean algorithm that all  $d_k \in \mathbb{Q}$ . Substituting  $z = c$  we get the required condition.  $\square$

### 3.3. Proof of Theorem 1.6

The proof follows the ideas of §3.2 and §5.3.

*Proof of Theorem 1.6.* (3)  $\implies$  (1) Analogously to the proof of Theorem 1.5(3)  $\implies$  (1).

(1)  $\implies$  (2). Suppose that a rectangle of ratio  $c$  is tiled by rectangles of ratios  $c$  and  $1/c$ . Rotating through  $\pi/2$  and stretching the figure we get a square tiled by squares and rectangles of ratio  $c^2$ . By Lemma 2.2 there exists an electrical circuit of conductance 1 with edge conductances 1 and  $c^2$ , in which all the edges are essential. Since there is at least one rectangle of ratio  $1/c$  in the initial tiling, it follows that the network contains at least one edge of conductance  $c^2$ . Replace each edge of conductance  $c^2$  (respectively, 1) in the network by an edge of conductance  $z \in \mathbb{C}$  (respectively,  $w \in \mathbb{C}$ ). Let  $C(z, w)$  the conductance of the obtained network. Denote by  $C(z) = C(z, 1)$ .

Let us prove that  $c^2$  is algebraic. Indeed, by Lemma 2.3(4) we have  $C'(c^2) > 0$  because there is at least one essential edge of conductance  $c^2$  in the network. Thus  $C(z)$  is nonconstant. By Lemma 2.3(1) it follows that  $C(z) \in \mathbb{Q}(z)$ . Since  $C(c^2) = 1$  it follows that  $c^2$  is algebraic.

Let  $z$  be an algebraic conjugate of  $c^2$  distinct from  $c^2$  itself. Then  $C(z, 1) = C(c^2) = 1$ .

Let us prove that  $z$  is a negative real number. First assume  $\text{Im } z < 0$ . Then  $\text{Re } iz > 0$ . By Lemma 2.3(2) it follows that  $\text{Re } C(iz, i) = \text{Re}(iC(z, 1)) = \text{Re } i = 0$ . Since  $C(iz, i)$  is a rational function it follows that any neighborhood of  $iz$  contains a point  $z'$  such that  $\text{Re } C(z', i) < 0$ . Taking sufficiently small neighborhood we get  $\text{Re } z' > 0$  because  $\text{Re } iz > 0$ . By continuity a neighborhood of  $i$  contains a point  $w'$  such that  $\text{Re } w' > 0$  and still  $\text{Re } C(z', w') < 0$ . The obtained inequalities contradict to Lemma 2.3(5). Case  $\text{Im } z > 0$  is violated similarly. Assume now  $z > c^2$ . Then by Lemma 2.3(4) we have  $1 = C(z) > C(c^2) = 1$ , a contradiction. Case  $0 \leq z < c^2$  is violated similarly. Thus  $z < 0$ .

(2)  $\implies$  (3) Let  $p(z)$  be a minimal polynomial of  $c^2$ . Since the roots of a minimal polynomial are all simple it follows that  $p(z^2) = (z^2 - c^2) \prod_{k=1}^n (z^2 + b_k^2)$  for some  $b_1 > \dots > b_n > 0$ . Take a polynomial  $q(z)$  with rational coefficients such that  $q(z) = z \prod_{k=1}^n (z^2 + a_k^2)$ , where  $a_1 > b_1 > a_2 > \dots > b_n > 0$ . Consider the odd rational function  $C(z) = q(z)/(zq(z) - p(z^2))$ . We have  $C(c) = 1/c$ .

Let us check that the function  $C(z)$  satisfies condition (3) of Lemma 2.5. The roots of  $C(z)$  are the numbers  $0, \pm ia_1, \dots, \pm ia_n$ . A direct evaluation shows that for each  $l = 1, \dots, n$

$$C'(\pm ia_l) = -\frac{q'(\pm ia_l)}{p(-a_l^2)} = \frac{2a_l^2}{(c^2 + a_l^2)(a_l^2 - b_l^2)} \prod_{k \neq l} \frac{a_k^2 - a_l^2}{b_k^2 - a_l^2} > 0$$

by the assumption  $a_1 > b_1 > a_2 > \dots > b_n > 0$ . Analogously  $C'(0) = -q'(0)/p(0) > 0$ .

Then by Lemma 2.5(3)  $\implies$  (5) the function  $C(z)$  satisfies condition (5) of Lemma 2.5. Since  $C(z) \in \mathbb{Q}(z)$  it follows by Euclidean algorithm that all  $d_k \in \mathbb{Q}$ . Substituting  $z = c$  we get the required condition.  $\square$

### 3.4. Remarks to main results

Let us define inductively a *series-parallel* electrical network. By definition, a network consisting of a single edge is series-parallel. If  $a$  and  $b$  are two series-parallel networks then both their series and parallel "unions" (see figure 1) are series-parallel.

**Theorem 3.1.** [5] *If a function  $C(c_1, c_2)$  satisfies conditions (1)–(3) of Theorem 1.3 then  $C(c_1, c_2)$  is the conductance of a series-parallel electrical network with edge conductances  $c_1$  and  $c_2$ .*



*Proof.* By conditions (1)–(3) of Theorem 1.3 we have  $C(c_1, c_2) = \sqrt{c_1 c_2} C(z, 1/z)$ , where  $z = \sqrt{c_1/c_2}$ , and the function  $C(z) = C(z, 1/z)$  satisfies condition (1) of Lemma 2.5. By Lemma 2.5(1)  $\implies$  (5) it satisfies condition (5). Therefore, say, for  $m$  even and  $C(0) = 0$ ,

$$C(c_1, c_2) = d_1 c_1 + \frac{1}{\frac{d_2}{c_2} + \frac{1}{d_3 c_1 + \cdots + \frac{d_m}{c_2}}}.$$

All the numbers  $d_k \in \mathbb{Q}$  by the Euclidean algorithm. Now the required series-parallel network is constructed analogously to the proof of Theorem 1.5(3)  $\implies$  (1).  $\square$

**Example 3.2.** A generalization of Theorem 3.1 to the case of 3 variables  $c_1, c_2, c_3$  is not true. E.g., consider the network with 4 vertices and edge conductances  $c_{13} = c_1, c_{23} = c_2, c_{24} = c_1, c_{14} = c_2, c_{34} = c_3$ . By Lemma 2.3(4) and a symmetry argument it follows that  $\partial C(c_1, c_2, c_3)/\partial c_3 = 0$  if  $c_1 = c_2$ . So  $C(c_1, c_2, c_3)$  cannot be the conductance of a series-parallel network, because all the edges of such networks are essential.

## 4. Variations

### 4.1. Tilings of polygons by rectangles

In this subsection we study the following problem.

**Problem 4.1.** Which polygons can be tiled by rectangles of given ratios  $c_1, \dots, c_n$ ?

Case  $n = 1, c_1 = 1$  of the problem is a description of polygons which can be tiled by squares, a problem posed in [15]. In case of hexagons such a description was obtained by R. Kenyon [21]. We give such description for a wide class of polygons.

Hereafter  $P$  is an *orthogonal* polygon, i.e., a polygon with sides parallel to coordinate axes. Assume that  $P$  is *simple*, i.e., the boundary  $\partial P$  has one connected component. Enumerate the sides parallel to the  $x$ -axis counterclockwise in  $\partial P$ . Let  $I_u$  be the *signed length* of the side  $u$ , where the sign of  $I_u$  is "+" ("−") if the  $P$  locally lies below (above) the side  $u$ . Let  $U_u$  be the  $y$ -coordinate of the side  $u$ . Assume that  $P$  is *generic*, i.e., the numbers  $U_1, \dots, U_b$  are pairwise distinct.

We need the following notion [9]. A sequence of boundary vertices  $(p_1, \dots, p_k, q_1, \dots, q_k)$  of a planar network is *circular*, if the sequence  $(p_1, \dots, p_k, q_k, \dots, q_1)$  is in counterclockwise order in the boundary of the unit disc. Denote by  $\Omega_b$  the set of real  $b \times b$  matrices  $C_{uv}$  satisfying the following properties:

- $C_{uv}$  is symmetric;
- the sum of the entries of  $C_{uv}$  in each row is zero;
- if  $(p_1, \dots, p_k, q_1, \dots, q_k)$  is a circular sequence then  $(-1)^k \det\{C_{p_i q_j}\}_{i,j=1}^k \geq 0$ .

**Theorem 4.2.** *Let  $P$  be a generic orthogonal polygon with  $b$  horizontal sides having signed lengths  $I_1, \dots, I_b$  and  $y$ -coordinates  $U_1, \dots, U_b$ . Then the following two conditions are equivalent:*

- (1) *the polygon  $P$  can be tiled by squares;*
- (2) *there is a matrix  $C_{uv} \in \Omega_b$  with rational entries such that  $I_v = \sum_{u=1}^b C_{uv} U_u$  for each  $v = 1, \dots, b$ .*

Cases  $b = 2$  and  $b = 3$  of this theorem are equivalent to Theorem 1.1 and [21, Theorem 9], respectively. Theorem 4.2 is algorithmic in the particular case when  $U_1, \dots, U_b$  are linearly independent over  $\mathbb{Q}$ . Proof of the theorem is constructive, i.e., gives an algorithm to construct the required tiling if the latter exists. Theorem 4.2 does not necessarily hold for *nongeneric* polygons, e.g., for an orthogonal polygon with

$$U_1 = U_3 = 0, \quad U_2 = 2, \quad U_4 = -4, \quad I_1 = \sqrt{2}, \quad I_2 = 2, \quad I_3 = 2 - \sqrt{2}, \quad I_4 = -4.$$

We also give a short proof of the following result:

**Theorem 4.3.** [28] *A generic orthogonal polygon with rational vertices can be tiled by rectangles of ratios  $c$  and  $1/c$  if and only if a square can be tiled by rectangles of ratios  $c$  and  $1/c$ .*

#### 4.2. Electrical impedance tomography

Our approach to Problem 4.1 follows the idea of [21, 7] and uses electrical networks with several terminals.

Hereafter we allow electrical circuits to have several boundary vertices  $1, \dots, b$  with prescribed voltages  $U_1, \dots, U_b$ . If an electrical circuit is planar, we assume that the boundary vertices are enumerated counterclockwise along the boundary of the unit disc. We do *not* assume that an electrical circuit is connected but require that each connected component contains a boundary vertex. The voltages and currents in such circuits are defined by the Ohm and the Kirchhoff current laws (C) and (I) from §2.

Consider the linear map  $\mathbb{C}^b \rightarrow \mathbb{C}^b$  which takes the vector of voltages  $(U_1, \dots, U_b)$  to the vector of *incoming currents*  $(I_1, \dots, I_b) = (\sum_{k=1}^n I_{1k}, \dots, \sum_{k=1}^n I_{bk})$  flowing inside the network through the vertices  $1, \dots, b$ , respectively. The matrix  $C_{uv}$  of this linear map is called the *response* of the network. This matrix is symmetric [9].

We reduce the results of §4.1 to the following problems even more interesting in themselves:

- *Direct problem.* Describe possible responses of electrical networks.
- *Inverse problem.* Describe possible networks having a given response.

These problems are solved for planar direct-current networks [9, 6, 7, 23]. Let us state certain deep results of Y. Colin de Verdière, E.B. Curtis and J.A. Morrow.

**Theorem 4.4.** [9, 8, 7, Theorem 5] *The set of all possible responses of planar electrical networks with  $b$  boundary vertices and positive edge conductances is the set  $\Omega_b$ .*

An electrical network is *minimal* (or *critical*) if it has minimal number of edges among all planar electrical networks with positive edge conductances and with the same response. The minimality of a network depends only on its graph [7]. In [9, 8, §9] an algorithm for finding edge conductances in a minimal network with given response is presented. This algorithm implies the following result.

**Theorem 4.5.** [9, §6.4] *Conductances of the edges in a minimal electrical network are uniquely determined by the response of the network. Each edge conductance is a rational function with rational coefficients in the entries of the response.*

For alternating-current circuits the direct problem is probably open. Let us state some basic results. The rest of §4 is not used in the proof of the above results.

**Theorem 4.6.** For  $b = 2$  or  $b = 3$  the following 2 conditions are equivalent:

- (1)  $C_{uv}$  is the response of a connected electrical network with  $b$  boundary vertices and with edge conductances having positive real parts;
- (2)  $C_{uv}$  is a complex  $b \times b$  matrix has the following 4 properties:
  - $C_{uv}$  is symmetric;
  - the sum of the entries of  $C_{uv}$  in each row is zero;
  - $\operatorname{Re} C_{uv}$  is non-negatively definite;
  - if  $\sum_{1 \leq u, v \leq b} \operatorname{Re} C_{uv} U_u U_v = 0$  then  $U_1 = \dots = U_b$ .

**Problem 4.7.** Does this result remain true for arbitrary  $b \geq 4$ ?

Unlike direct-current networks *nonboundary vertices in alternating-current networks can be detected by the response*. For instance, by Theorem 4.6 there are electrical networks with response  $\begin{pmatrix} 2 & 1 & -3 \\ 1 & 2 & -3 \\ -3 & -3 & 6 \end{pmatrix}$ ; any such network necessarily has nonboundary vertices.

#### 4.3. Random walks

A *random walk* on an electrical network (or on a *weighted graph*) is the Markov chain with the transition matrix  $P_{kl} = c_{kl} / \sum_{j=1}^n c_{jk}$ . Such Markov chain is ergodic and reversible. Denote by  $k_1 l_1, \dots, k_m l_m$  all the edges of the Markov chain. The following theorem allows to translate the results of §1–§2 to the language of random walks.

**Theorem 4.8.** [11, page 42] Let  $P(c_{k_1 l_1}, \dots, c_{k_m l_m})$  be the probability that a random walk starting at vertex 1 reaches vertex 2 before returning to 1. Let  $C(c_{k_1 l_1}, \dots, c_{k_m l_m})$  be the conductance of the network (with boundary vertices 1 and 2). Then  $P(c_{k_1 l_1}, \dots, c_{k_m l_m}) = C(c_{k_1 l_1}, \dots, c_{k_m l_m}) / (c_{12} + \dots + c_{1n})$ .

For instance, a translation of Lemmas 2.3(1) and (5) is:

**Corollary 4.9.** The probability  $P(c_{k_1 l_1}, \dots, c_{k_m l_m})$  is a rational function in  $c_{k_1 l_1}, \dots, c_{k_m l_m}$ . If  $\operatorname{Re} c_{k_1 l_1}, \dots, \operatorname{Re} c_{k_m l_m} > 0$  then  $\operatorname{Re} ((c_{12} + \dots + c_{1n}) P(c_{k_1 l_1}, \dots, c_{k_m l_m})) > 0$ .

The latter result does not necessarily hold for *nonreversible* Markov chains, e.g., for a Markov chain with vertices 1, 2, 3, 4 and oriented edges 14, 42, 43.

Nonreversible planar Markov chains have a geometric interpretation as tilings of trapezoids by trapezoids [21]. Here a *trapezoid* is a 4-gon with two sides parallel to the  $x$ -axis. The *ratio* of the trapezoid is the length of the horizontal middle edge divided by the height. Natural problems are: *generalize the results of the paper to tilings by trapezoids; infinite tilings; signed tilings*.

## 5. Generalization of main ideas

### 5.1. Electrical circuits

Our approach is based on a generalization of the results of §2 to electrical circuits with  $b$  terminals. Short proofs of the results of §2 are obtained in this section as particular case  $b = 2$ . Our proof of Lemma 5.2(3), generalizing Lemma 2.3(3), is probably new. All the proofs are based on the following fundamental *energy conservation law*.

**Claim 5.1.** Let  $E(U, I)$  be a bilinear function. Consider an electrical network with the vertices  $1, \dots, n$  such that  $1, \dots, b$  are the boundary ones. Suppose that the numbers  $U_k$ ,  $1 \leq k \leq n$ , and  $I_{kl}$ ,  $1 \leq k, l \leq n$ , satisfy laws (C),(I) from §2. Set  $I_u = \sum_{k=1}^n I_{uk}$ . Then

$$\sum_{1 \leq k < l \leq n} E(U_k - U_l, I_{kl}) = \sum_{1 \leq u \leq b} E(U_u, I_u).$$

We usually apply this claim for the *energy dissipation* function  $E(U, I) = \operatorname{Re}(U\bar{I})$ .

*Proof of Claim 5.1.* By law (C) we have  $I_{lk} = -I_{kl}$ . Hence by law (I) we have

$$\sum_{1 \leq k < l \leq n} E(U_k - U_l, I_{kl}) = \sum_{k=1}^n E(U_k, \sum_{l=1}^n I_{kl}) = \sum_{1 \leq u \leq b} E(U_u, I_u).$$

□

Let us prove Theorem 2.1 for electrical circuits with  $b$  boundary vertices and with complex edge conductances having positive real part.

*Proof of Theorem 2.1. Uniqueness.* Suppose there are two collections of currents  $I_{kl}^{I, II}$  and voltages  $U_k^{I, II}$  satisfying laws (C),(I). Then their difference  $I_{kl} = I_{kl}^I - I_{kl}^{II}$ ,  $U_k = U_k^I - U_k^{II}$  satisfies (C),(I) for zero incoming voltages  $U_1 = \dots = U_b = 0$ . Then by Claim 5.1 we have

$$\sum_{1 \leq k < l \leq n} \operatorname{Re} \bar{c}_{kl} |U_k - U_l|^2 = \sum_{1 \leq k < l \leq n} \operatorname{Re}((U_k - U_l)\bar{I}_{kl}) = \sum_{1 \leq u \leq b} \operatorname{Re}(U_u \bar{I}_u) = 0.$$

Here for each  $k, l$  either  $\operatorname{Re} c_{kl} > 0$  or  $c_{kl} = 0$ . Thus each  $\operatorname{Re} \bar{c}_{kl} |U_k - U_l|^2 = 0$ . Since all the connected components of the circuit contain boundary vertices it follows that all  $U_k$  are equal. Hence each  $U_k = 0$ ,  $I_{kl} = 0$  and thus each  $I_{kl}^I = I_{kl}^{II}$ ,  $U_k^I = U_k^{II}$ .

*Existence.* The number of linear equations in the system (C),(I) equals the number of variables. By the previous paragraph the system has a unique solution for  $U_1 = \dots = U_b = 0$ . Thus by the finite-dimensional Fredholm alternative it has a solution for any  $U_1, \dots, U_b$ . □

The following result generalizes Lemma 2.3.

**Lemma 5.2.** Suppose that an electrical network has  $b$  boundary vertices and  $m$  edges of conductances  $c_1, \dots, c_m$ . Then the response of the network  $C_{uv}(c_1, \dots, c_m)$  has the following properties:

- (1)  $C_{uv}(c_1, \dots, c_m) \in \mathbb{Q}(c_1, \dots, c_m)^{b \times b}$ ;
- (2)  $C_{uv}(c_1, \dots, c_m)$  is degree 1 homogeneous;
- (3)  $\frac{\partial}{\partial c_j} C_{uv}(c_1, \dots, c_m) = (V_{ku} - V_{lu})(V_{kv} - V_{lv})$ , where  $k$  and  $l$  are the endpoints of the edge  $j$  and  $V_{pq}$  is the matrix of the linear map  $(U_1, \dots, U_b) \mapsto (U_1, \dots, U_n)$ ;
- (4) if  $c_1, \dots, c_m > 0$  then  $\frac{\partial}{\partial c_j} C_{uv}(c_1, \dots, c_m)$  is non-negatively definite;
- (5) if  $\operatorname{Re} c_1, \dots, \operatorname{Re} c_m > 0$  then  $\operatorname{Re} C_{uv}(c_1, \dots, c_m)$  is non-negatively definite.

*Proof of Lemma 5.2.* (1) By Theorem 2.1 and the Crammer rule the solution  $\{I_{kl}(U_1, \dots, U_b)\}$  of the system of linear equations (C), (I) consists of linear functions in  $U_1, \dots, U_b$  with coefficients being rational functions in  $c_1, \dots, c_m$ . So the entries of the matrix of the linear map  $(U_1, \dots, U_b) \mapsto \sum_{k=1}^n I_{uk}(U_1, \dots, U_b)$  are rational functions in  $c_1, \dots, c_m$ .

(2) Consider the system of linear equations obtained from laws (C), (I) by substituting  $tc_1, \dots, tc_m$  for  $c_1, \dots, c_m$ . It defines the same voltages as the initial one and the currents are scaled by  $t$ . So  $C(tc_1, \dots, tc_m) = tC(c_1, \dots, c_m)$ .

(3) Set  $E(U, I) = \frac{\partial U}{\partial c_{kl}} I - U \frac{\partial I}{\partial c_{kl}}$ . Then  $E(U_k - U_l, I_{kl}) = (U_k - U_l)^2$  and  $E(U_p - U_q, I_{pq}) = 0$  for  $pq \neq kl$ . Thus by Claim 5.1 we have

$$\begin{aligned} \sum_{1 \leq u, v \leq b} \frac{\partial C_{uv}}{\partial c_{kl}} U_u U_v &= \sum_{1 \leq u \leq b} E(U_u, I_u) = \sum_{1 \leq p < q \leq n} E(U_p - U_q, I_{pq}) = \\ &= (U_k - U_l)^2 = \sum_{1 \leq u, v \leq b} (V_{ku} - V_{lu})(V_{kv} - V_{lv}) U_u U_v. \end{aligned}$$

(4) This follows directly from the latter formula.

(5) Assume that for each  $k, l$  either  $\operatorname{Re} c_{kl} > 0$  or  $c_{kl} = 0$ . Take  $U_1, \dots, U_b \in \mathbb{R}$ . By Claim 5.1 we have

$$\begin{aligned} \sum_{1 \leq u, v \leq b} \operatorname{Re} C_{uv} U_u U_v &= \sum_{1 \leq u \leq b} \operatorname{Re}(U_u \bar{I}_u) = \\ &= \sum_{1 \leq k < l \leq n} \operatorname{Re}((U_k - U_l) \bar{I}_{kl}) = \sum_{1 \leq k < l \leq n} \operatorname{Re} c_{kl} |U_k - U_l|^2 \geq 0. \end{aligned}$$

□

*Remark 5.3.* If the network is connected then the latter inequality is strict unless  $U_1 = \dots = U_b$ .

## 5.2. Tilings and networks

Part (2)  $\implies$  (1) of the following result is probably new, cf. [1, 21].

**Lemma 5.4.** *Let  $P$  be a generic orthogonal polygon with horizontal sides of signed lengths  $I_1, \dots, I_b$  and  $y$ -coordinates  $U_1, \dots, U_b$ . Then the following 2 conditions are equivalent:*

- (1) *the polygon  $P$  can be tiled by  $m$  rectangles of ratios  $c_1, \dots, c_m$ ;*
- (2) *there is a planar electrical circuit with  $b$  boundary vertices,  $m$  essential edges of conductances  $c_1, \dots, c_m > 0$ , incoming voltages  $U_1, \dots, U_b$  and incoming currents  $I_1, \dots, I_b$ .*

*Remark 5.5.* Condition (2) itself does not guarantee the existence of a rectangular polygon with horizontal sides of signed lengths  $I_1, \dots, I_b$  and  $y$ -coordinates  $U_1, \dots, U_b$ . Lemma 5.4(1)  $\implies$  (2) is not necessarily true for *nongeneric* polygons.

*Proof of Lemma 5.4.* (1)  $\implies$  (2). Take a generic polygon  $P$  tiled by rectangles.

Let us construct the graph of the required network, see figure 3. Consider the union of the horizontal sides of all rectangles of the tiling. This union splits into several disjoint segments called *horizontal cuts*. Paint red (bold) all horizontal cuts except small neighborhoods of their endpoints. Paint blue (dashed) the vertical centerline of each rectangle in the tiling.

Contract all red segments. Then the blue set "becomes" a graph  $G$  and the polygon  $P$  "becomes" a topological disc  $D$  (since the  $y$ -coordinates of the horizontal sides of  $P$  are distinct it follows that each red segment has not more than one common point with  $\partial P$ ). Denote by  $1, \dots, b$  the vertices of the graph  $G$  obtained from the red segments in the horizontal cuts containing the sides of  $P$  and by  $b + 1, \dots, n$  — the other vertices.

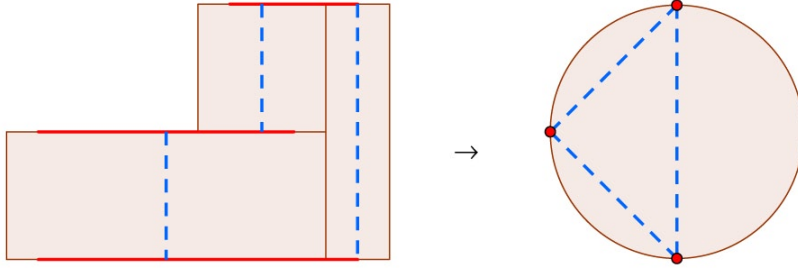


Figure 3: Construction of an electrical network

Clearly,  $G \subset D$ ,  $G \cap \partial D = \{1, \dots, b\}$  and each connected component of  $G$  contains a boundary vertex. Thus  $G$  is a graph of a planar network.

Let us define the voltages, currents and conductances in the network. For each vertex  $k = 1, \dots, n$  of the graph  $G$  set  $U_k$  to be the  $y$ -coordinate of the horizontal red segment contracted to the vertex. For each edge  $kl$  of the graph  $G$ , obtained from the vertical centerline of a rectangle in the tiling, set  $I_{kl}$  and  $c_{kl}$  to be the horizontal side (with an appropriate sign) and the ratio of the rectangle, respectively. The laws (C), (I) are now checked directly. The constructed network is the required.

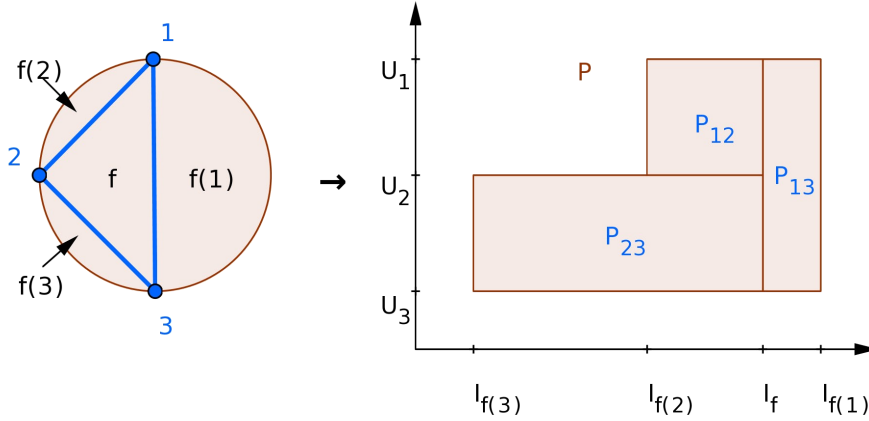


Figure 4: Construction of a tiling

(2)  $\implies$  (1). Take an electrical network as in (2). Construct a tiling of  $P$  as follows.

Let  $e$  be an edge of the network. Denote by  $e \uparrow$  ( $e \downarrow$ ) the endpoint of  $e$  with higher (lower) voltage (it is well-defined by the assumption that all the edges are essential). By a *face* we mean a connected component of the complement to the network in the unit disc  $D$ . Denote by  $e \leftarrow$  ( $e \rightarrow$ ) the face that borders the edge  $e$  from the left-hand (right-hand) side while one moves along the edge  $e$  from  $e \uparrow$  to  $e \downarrow$ .

By law (I) it follows that to each face  $f$  one can assign a number  $I_f$  in such a way that  $I_{kl\leftarrow} - I_{kl\rightarrow} = I_{kl}$ . Without loss of generality assume  $\min_f I_f = \min_{(x,y) \in P} x$ , where the minimum in the left-hand side is over all the faces  $f$  meeting  $\partial D$ .

Let  $P_e$  be the rectangle with the vertices  $(I_{e\rightarrow}, U_{e\uparrow}), (I_{e\rightarrow}, U_{e\downarrow}), (I_{e\leftarrow}, U_{e\uparrow}), (I_{e\leftarrow}, U_{e\downarrow})$ . The rectangles  $P_e$ , where  $e$  runs through all the edges of the network, tile the polygon  $P$  by the following two claims ( $P_e$ -s cover  $P$  by Claim 5.6 and do not overlap by Claim 5.7).  $\square$



**Claim 5.6.**  $\bigcup_e P_e = P$ .

*Proof.* It suffices to prove that  $\partial \bigcup_e P_e \subset \partial P$ . Since  $\partial P$  is a simple closed curve in the plane and  $\bigcup_e P_e$  is bounded, the claim will follow.

We need the following description of the boundary  $\partial P$ , see figure 4. Boundary vertices split  $\partial D$  into  $b$  arcs. Start from vertex  $b$  and move along the circle  $\partial D$  counterclockwise. Enumerate the arcs in the order they appear in the motion. Denote by  $f(v)$  the face containing the arc  $v$ . Denote by  $H_v$  the segment joining the points  $(I_{f(v)}, U_v)$  and  $(I_{f(v+1)}, U_v)$ . Denote by  $V_v$  the segment joining the points  $(I_{f(v)}, U_{v-1})$  and  $(I_{f(v)}, U_v)$ , where we set  $U_0 = U_b$ . Clearly,  $\partial P = \bigcup_{v=1}^b (H_v \cup V_v)$ .

Take a "generic" point  $p \in \partial \bigcup_e P_e$ , say, in a horizontal side of the "polygon"  $\bigcup_e P_e$ . The point  $p$  necessarily belongs to a horizontal side of a rectangle in the tiling, say, to the top side of a rectangle  $P_e$ . Denote by  $v = e \uparrow$  the vertex of  $e$  of higher voltage.

Draw a horizontal line  $H$  through the top side of the rectangle  $P_e$ . We say that a rectangle  $P_d$  is *adjacent* if the vertex  $v$  is an endpoint of the edge  $d$ . Adjacent rectangles border upon the line  $H$  either from above or from below.

First assume that  $v$  is nonboundary. A simple induction shows that each point of  $H$  (except a finite set) is bordered by the same number of adjacent rectangles  $P_d$  from above and from below. Since the rectangle  $P_e$  borders upon the point  $p$  from below and  $p$  is "generic" it follows that some adjacent rectangle  $P_d$  borders upon it from above. Thus  $p$  belongs to  $\text{Int } P_e \cup P_f \subset \text{Int } \bigcup_e P_e$ , a contradiction.

So  $v$  is a boundary vertex. Analogously to the above each point of  $H - H_v$  (except a finite set) is bordered by the same number of adjacent rectangles  $P_d$  from above and from below. Hence  $p \in H_v$  and thus  $p \in \partial P$ .  $\square$

**Claim 5.7.**  $\sum_e \text{Area}(P_e) = \text{Area}(P)$ .

*Proof.* This follows immediately from Claim 5.1 because  $\text{Area}(P_{kl}) = (U_k - U_l)I_{kl}$  and  $\text{Area}(P) = \sum_{1 \leq u \leq b} U_u I_u$ .  $\square$

### 5.3. Positive real functions

Let us prove Lemma 2.5. For a generalization to the case  $b > 2$  see [12].

*Proof of Lemma 2.5.* (1)  $\implies$  (2). Indeed, if  $\text{Re } z \leq 0$  then  $\text{Re } C(z) = -\text{Re } C(-z) \leq 0$  and thus  $C(z) \neq 1$ .

(2)  $\implies$  (1). Consider the equation  $C(z) = w$ . Move  $w$  continuously in the half-plane  $\text{Re } w > 0$ . The roots cannot cross the line  $\text{Re } z = 0$  (because  $\text{Re } z = 0$  implies  $\text{Re } C(z) = 0$  for an odd function  $C(z) \in \mathbb{R}(z)$ ). Thus for each  $w$  in the half-plane  $\text{Re } w > 0$  all roots of  $C(z) = w$  are in the half-plane  $\text{Re } z > 0$ . Since  $C(z)$  is odd it follows that the same is true for the half-planes  $\text{Re } w < 0$ ,  $\text{Re } z < 0$ . So (1) holds.

(1)  $\implies$  (3). Suppose that  $C(z) = 0$ , where  $z \in \mathbb{C}$ . Then  $\text{Re } z = 0$  because  $\text{Re } z > 0 \implies \text{Re } C(z) > 0$  and  $\text{Re } z < 0 \implies \text{Re } C(z) = -\text{Re } C(-z) < 0$ . Since condition (1) and its converse hold in a neighborhood of the point  $z$  it follows that  $C'(z) > 0$ . A simple limiting argument proves the same for  $z = \infty$ .

(3)  $\implies$  (4) Assume for simplicity that  $C(\infty) \neq 0$ . Let  $z_1, \dots, z_m$  be the roots of  $C(z)$ . Since  $C'(z_k) > 0$  it follows that the roots are simple. Thus  $C(z)$  has not more than  $m$  poles. The roots split the projective line  $\text{Re } z = 0$  into  $m$  "segments". Since  $C'(z_k) > 0$  it follows that for sufficiently small  $\epsilon > 0$  we have  $C(z_k - i\epsilon) < 0$  and  $C(z_k + i\epsilon) > 0$ . By intermediate value theorem it follows that each of the segments contains a pole of  $C(z)$ . Thus all the  $m$  poles of  $C(z)$  belong to the line  $\text{Re } z = 0$  and alternate with the roots. So (4) holds.

(4)  $\implies$  (5). Denote by  $ht C(z)$  the sum of the degrees of the nominator and the denominator of  $C(z)$ . The proof is by induction over  $ht C(z)$ . If  $ht C(z) = 1$  then there is nothing to prove. Assume that, say,  $C(z)$  equals the expression from condition (4), where  $n \geq 1$  and  $b_n \neq 0$ .

Denote by  $r(z) = 1/(C(z) - d_1 z)$  and  $q(z) = 1/C(z)$ . Let us prove that  $r(z)$  satisfies condition (3). Indeed, the roots of  $r(z)$  are the numbers  $\pm ib_1, \dots, \pm ib_n$ . For each  $l = 1, \dots, n$

$$r'(\pm ib_l) = q'(\pm ib_l) = \frac{2}{d_1(a_l^2 - b_l^2)} \prod_{k \neq l} \frac{b_k^2 - b_l^2}{a_k^2 - b_l^2} > 0$$

by the condition  $a_1 > b_1 > a_2 > \dots > b_n \geq 0$ .

Hence by Lemma 2.5(3)  $\implies$  (4) it follows that  $r(z)$  satisfies condition (4) as well. On the other hand  $ht r(z) < ht C(z)$ . By inductive hypothesis,  $r(z)$  satisfies condition (5). Thus  $C(z) = 1/(d_1 z + r(z))$  also satisfies condition (5).

(5)  $\implies$  (1). This follows by a simple induction over  $m$ .  $\square$

## 6. Proof of variations

### 6.1. Proof of Theorem 4.2

*Proof of Theorem 4.2.* (1)  $\implies$  (2). Let the polygon  $P$  be tiled by squares. By Lemma 5.4 there is a planar electrical circuit with edge conductances 1, incoming voltages  $U_1, \dots, U_b$  and incoming currents  $I_1, \dots, I_b$ . Let  $C_{uv}$  be the response of the circuit. Then  $I_v = \sum C_{uv} U_u$ . By Lemma 5.2(1) all the entries of  $C_{uv}$  are rational. By Theorem 4.4 we have  $C_{uv} \in \Omega_b$ .

(2)  $\implies$  (1). Let  $C_{uv} \in \Omega_b$  be a matrix with rational entries such that  $I_v = \sum C_{uv} U_u$ . By Theorem 4.4 there are planar electrical networks with the response  $C_{uv}$ . Take a minimal network with this property. By Theorem 4.5 the conductances of all the edges of the network are rational. Set the incoming voltages to be  $U_1, \dots, U_b$ . Then the incoming currents are  $I_1, \dots, I_b$ . Delete all unessential edges from the circuit. By Lemma 5.4 it follows that the polygon  $P$  can be tiled by rectangles of rational ratio, and hence by squares.  $\square$

**Corollary 6.1.** (of Lemmas 5.2, 5.4 and Theorem 4.4) *If a generic orthogonal polygon  $P$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$  then there is a function  $C_{uv}(z_1, \dots, z_n)$  satisfying conditions (1), (2) and (5) of Lemma 5.2 such that  $C(c_1, \dots, c_n) \in \Omega_b$  and  $I_v = \sum_{1 \leq u \leq b} C_{uv}(c_1, \dots, c_n) U_u$  for each  $v = 1, \dots, b$ .*

### 6.2. Proof of Theorem 4.3

*Proof of Theorem 4.3.*  $\Leftarrow$ . This holds because a polygon with rational vertices can be tiled by squares.

$\implies$ . Suppose that  $P$  can be tiled by rectangles of ratios  $c$  and  $1/c$ . Let us prove analogously to the proof of Theorem 1.5(1)  $\implies$  (2) that all algebraic conjugates of  $c$  have positive real parts. Then Theorem 4.3 will follow from Theorem 1.5(2)  $\implies$  (1).

Consider the circuit given by Lemma 5.4. Replace each edge of conductance  $c$  (respectively,  $1/c$ ) in the circuit by an edge of conductance  $z \in \mathbb{C}$  (respectively,  $1/z$ ). Let  $C_{uv}(z)$  be the response of the obtained circuit. Consider the *energy dissipation function*  $E(z) = \sum_{1 \leq u, v \leq b} C_{uv}(z) U_u U_v$ . Since each  $U_u \in \mathbb{Q}$  it follows by Lemma 5.2(1) that  $E(z) \in \mathbb{Q}(z)$ . Clearly,  $E(c) = \sum_{1 \leq u \leq b} I_u U_u = \text{Area}(P)$ . Thus  $E(c) \in \mathbb{Q}$  and  $E(c) > 0$ .

Since  $E(z) \in \mathbb{Q}(z)$  and  $E(c) \in \mathbb{Q}$  it follows that  $c$  is algebraic ( $E(z)$  is nonconstant because  $E(-c) = -E(c) < 0$  by Lemma 5.2(2)). Let  $z$  be an algebraic conjugate of  $c$ . Then  $E(z) = E(c) > 0$ .

Let us prove that  $\operatorname{Re} z > 0$ . Indeed, first assume that  $\operatorname{Re} z < 0$ . Then by Lemma 5.2(5) we have  $0 \leq \operatorname{Re} E(-z) = -\operatorname{Re} E(z) < 0$ , a contradiction. A simple limiting argument shows that assumption  $\operatorname{Re} z = 0$  also leads to a contradiction. Thus  $\operatorname{Re} z > 0$ .  $\square$

### 6.3. Proof of Theorem 4.6

*Proof of Theorem 4.6.* (1)  $\implies$  (2). This follows from Lemma 5.2(5) and Remark 5.3.

(2)  $\implies$  (1). For  $b = 2$  there is nothing to prove. Assume that  $b = 3$ . Let  $\delta > 0$  be a small number,  $r_{uv} = -\operatorname{Re} C_{uv} - \delta$ ,  $m_{uv} = -\operatorname{Im} C_{uv}$ . By the assumption of the theorem it follows that  $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$  is positively definite. Thus  $\begin{pmatrix} r_{31} + r_{12} & -r_{12} \\ -r_{12} & r_{12} + r_{23} \end{pmatrix}$  is positively definite for sufficiently small  $\delta$ . Hence  $r_{12} + r_{23}, r_{31} + r_{12}, r_{12}r_{23} + r_{23}r_{31} + r_{31}r_{12} > 0$ . Analogously  $r_{23} + r_{31} > 0$ . Thus at least two of the numbers  $r_{12}, r_{23}, r_{31}$  are positive.

If  $r_{12}, r_{23}, r_{31} > 0$  then the required network is a triangle 123 with edge conductances  $c_{kl} = r_{kl} + im_{kl} + \delta$ .

Now assume that exactly one of the numbers  $r_{12}, r_{23}, r_{31}$ , say,  $r_{31}$  is nonpositive. Take a large number  $M$  and denote by  $\Delta_M = r_{12}r_{23} + r_{23}r_{31} + r_{31}r_{12} + iM(r_{23} + r_{12})$ . The required network is a complete graph on the vertices 1, 2, 3, 4 with edge conductances

$$\begin{aligned} c_{12} &= im_{12} + \delta, & c_{14} &= \Delta_M / r_{23}, \\ c_{23} &= im_{23} + \delta, & c_{34} &= \Delta_M / r_{12}, \\ c_{31} &= im_{31} + \delta - iM, & c_{24} &= \Delta_M / (r_{31} + iM). \end{aligned}$$

Clearly, for  $M^2 > (r_{12}r_{23} + r_{23}r_{31} + r_{31}r_{12})|r_{31}|/(r_{23} + r_{12})$  we have all  $\operatorname{Re} c_{kl} > 0$ .

Let us show by electrical transformations that the network has response  $C_{uv}$ . Indeed, replace the "letter Y" formed by the edges 14, 24 and 34 by a "triangle  $\Delta$ " formed by 3 new edges of conductances  $c'_{12} = r_{12}$ ,  $c'_{23} = r_{23}$  and  $c'_{31} = r_{31} + iM$ . This  $Y\Delta$ -transformation does not change the response [21, page 12]. The obtained network has 3 pairs of multiple edges. Thus it has the same response as a triangle with edge conductances  $r_{12} + im_{12} + \delta$ ,  $r_{23} + im_{23} + \delta$ ,  $r_{31} + im_{31} + \delta$ . So the network has the response  $C_{uv}$ .  $\square$

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